On the Relationship between Fuzzy and Boolean Cellular Automata *

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Abstract

Fuzzy cellular automata (FCA) are continuous cellular automata where the local rule is defined as the "fuzzification" of the local rule of a corresponding Boolean cellular automaton in disjunctive normal form. In this paper, we are interested in the relationship between Boolean and fuzzy models and, for the first time, we analytically show the existence of a strong connection between them by focusing on two properties: density conservation and additivity.

We begin by showing that the density conservation property, extensively studied in the Boolean domain, is preserved in the fuzzy domain: a Boolean CA is density conserving if and only if the corresponding FCA is sum preserving. A similar result is established for another novel "spatial" density conservation property. Second, we prove an interesting parallel between additivity of Boolean CA and oscillations of the corresponding fuzzy CA around its fixed point. In fact, we show that a Boolean CA is additive if and only if the behaviour of the corresponding fuzzy CA around its fixed point coincides with the Boolean behaviour. Finally, we give a probabilistic interpretation of our fuzzification which establishes an equivalence between convergent fuzzy CA and the mean field approximation on Boolean CA, an estimation of their asymptotic density.

These connections between the Boolean and the fuzzy models are the first formal proofs of a relationship between them.

Keywords. Cellular automata, continuous cellular automata, fuzzy cellular automata, density conservation, additivity.

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1 Introduction

1.1 Fuzzy Cellular Automata

Since the introduction of cellular automata (CA) by von Neumann [27] the study of their properties, in particular of Boolean CA, has interested various disciplines as diverse as ecology, biology, engineering and theoretical computer science (e.g., see [4, 10, 12, 17, 29]).

Fuzzy cellular automata (FCA) are a particular type of continuous cellular automata where the local transition rule is the “fuzzification" of the local rule of the corresponding Boolean cellular automaton in disjunctive normal form[1]. Fuzzy cellular automata were introduced in [7] and some of their properties have been studied in [13, 14, 21, 22], especially when considering finite configurations in quiescent backgrounds. Recently, they have been shown to be useful tools for pattern recognition purposes (e.g., see [19, 20]), and good models for generating images mimicking nature (e.g. [9, 26]).

To date, little is know about the dynamics of FCA, and the only existing results concern elementary FCA (i.e., with dimension and neighbourhood one). In quiescent backgrounds, it has been shown that none of the elementary FCA has chaotic dynamics [14, 21, 22]. The case of circular elementary FCA has been studied experimentally from random initial configurations. An empirical classification has been proposed based on these studies [13] suggesting that all elementary rules have asymptotic periodic behaviour but, surprisingly, with periods of only certain lengths: 1,2,4, and n (where n is the size of the circular lattice). Analytical studies to formally confirm the proposed classification have begun in [3].

In addition to the many interesting questions about the properties of fuzzy CA and their applications, a crucial research question is the nature of the relationship between fuzzy CA and Boolean CA. In fact, the dynamics of fuzzy CA might shed some light on their Boolean counter-parts, and properties of Boolean CA could be interpreted differently in light of those of fuzzy CA. If clear links between the two systems can be established, properties of Boolean CA not previously observed might be revealed by their presence in FCA. Unfortunately, until now, no such light had been shed and no such results existed. In fact, it was not even clear whether such a connection existed. To date, none of the studies on fuzzy asymptotic behaviour seemed to suggest any similarities between the two models. The only interesting link between them was observed in [14] for the case of elementary Boolean rule 90 (one of the most studied elementary CA rules) where it was shown that its asymptotic behaviour was identical to the dynamics of the oscillations of the corresponding fuzzy CA around its fixed point, one half. In other words, fuzzy rule 90 eventually stabilizes on one half, oscillating around it and the oscillations follow Boolean rule 90 itself. The reasons for such behaviour and the general implications for fuzzy CA were unknown until now.

[1]These are not to be confused with a variant of cellular automata, also called fuzzy cellular automata, where the fuzziness refers to the choice of a deterministic local rule (e.g., see [1])

1.2 Our Results

The main results of this paper are the formal proofs of the existence of a strong relationship between fuzzy and Boolean CA with respect to two properties: density conservation and additivity.

We begin the study of density with the exploration of *density conservation* in the discrete and continuous models. More precisely, we consider two types of density conservation: a temporal one, which is the classical notion of *number conservation* and has been studied extensively in the Boolean domain (e.g., see [5, 6, 11, 12, 24]), and a spatial one that has not been studied before. We prove that our fuzzification preserves both: in other words, a one-dimensional Boolean circular cellular automaton (i.e., with periodic initial configuration) is density-conserving if and only if its corresponding fuzzy circular cellular automaton is *sum preserving*. The result follows from the fact that DNF-fuzzification results in the unique extension to the Boolean rule which is affine in each variable. As a simple corollary of our result, we re-discover the number conservation property of elementary rule 184 (already well known in the Boolean domain) and we find an interesting spatial density conservation property of another elementary rule (rule 46) that can be translated into the Boolean domain: for any configuration of even size at time $t > 0$, the density of the odd cells is equal to the density of the even cells.

We continue by examining a class of fuzzy rules whose asymptotic behaviour continues to reflect that of their associated Boolean rules even as they converge to a fixed point. We call this property *self-oscillation*. We show that a fuzzy CA rule is self-oscillating if and only if the corresponding Boolean CA rule is an additive rule or its negation. This result fully characterizes the class of d-dimensional, infinite CA with this behaviour, thus explaining the phenomenon observed in [14] for rule 90.

Finally, we explore the unique nature of our fuzzification based on a probabilistic interpretation that links a fuzzy value in a given location during the evolution of a FCA with the probability of a one occurring in that location in the corresponding Boolean CA. We show that in the case of convergent fuzzy CA, the point of convergence is a stable density of the *mean field approximation* [16] of the corresponding Boolean CA, a well known estimate of its asymptotic density. Although for simplicity of description the rest of the paper takes its examples from one-dimensional CA, *all the results hold for any dimension* d.

2 Definitions

A d-dimensional infinite Boolean cellular automaton can be described by a quadruplet $C\langle \mathbb{Z}^d, \{0,1\}, N, g\rangle$ where: $\mathbb{Z}^d$ represents the set of *cells*, $\{0,1\}$ is the set of Boolean *states* of the cells, N is the *neighbourhood* of a cell and can be defined in different ways but usually contains the cell itself plus the neighbouring cells up to a certain radius, and $g : \{0,1\}^{|N|} \to \{0,1\}$ is the *local transition rule* (or simply *local rule*) of the automaton. Given an initial configuration, C^0, that is a mapping $C^0 : \mathbb{Z}^d \rightarrow \{0, 1\}$, cell states are synchronously updated at each time step by the local transition rule applied to their neighbourhoods. A configuration is the resulting map $C^t : \mathbb{Z}^d \rightarrow \{0, 1\}$ at any time t. A d-dimensional Boolean cellular automaton is said to have a *finite* configuration if it has a finite number of non-zero states in an infinite quiescent background. That is, $C^t(z) = 0$ for all but finitely many $z \in \mathbb{Z}^d$. *Circular* cellular automata can be thought of as infinite CA with a periodic repeating pattern, or as a finite circular d-dimensional grid.

In the case of *one-dimensional circular* Boolean cellular automata, a configuration is a finite vector $\mathbf{X}^t \in \{0, 1\}^n = (x_0^t, x_,^t \ldots, x_{n-1}^t)$ where cells are index modulo n, the length of the finite array. Alternatively, one can think of an infinite array containing a periodic configuration. The neighbourhood of a cell consists of the cell itself and its r left and right neighbours, thus the local transition rule has the form: $g : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$. The global dynamics of a one-dimensional circular cellular automaton composed of n cells is then defined by the global transition rule: $G : \{0, 1\}^n \rightarrow \{0, 1\}^n \quad s.t. \quad \forall \mathbf{X} \in \{0, 1\}^n, \forall i \in \{0, \ldots, n-1\}$, the i-th component $G(X)_i$ of $G(X)$ is $G(X)_i = g(x_{i-r}, \ldots, x_i, \ldots, x_{i+r})$, where all operations on indices are modulo n. Cellular automata with dimension and radius one are called *elementary*.

The local transition rule g of a Boolean CA is typically given in tabular form by listing the 2^{2r+1} binary tuples corresponding to the 2^{2r+1} possible local configurations a cell can detect in its direct neighbourhood, and mapping each tuple to a Boolean value b_i ($0 \leq i \leq 2^{2r+1} - 1$): $(00 \cdots 00, 00 \cdots 01, \ldots\ldots, 11 \cdots 10, 11 \cdots 11) \rightarrow (b_0, \cdots, b_{2^{2r+1}})$. The binary representation $(b_0, \cdots, b_{2^{2r+1}})$ is often converted into the decimal representation $\sum_i 2^i b_i$, and this value is typically used as the decimal code of the rule (or rule number). Let us denote by d_i the tuple mapping to b_i, and by $\mathcal{T}_1$ the set of tuples mapping to one. The local transition rule can also be canonically expressed in *disjunctive normal form* (DNF) as follows:

$$g(v_{-r}, \cdots, v_r) = \bigvee_{i < 2^{2r+1}} b_i \bigwedge_{j=-r:r} v_j^{d_{i,j+r}}$$

where $d_{i,j}$ is the j-th digit, from left to right of d_i (counting from zero) and v_j^0 (resp. v_j^1) stands for $\neg v_j$ (resp. v_j) i.e., $\bigwedge_{j=-r:r} v_j^{d_{i,j+r}}$ will be equal to one precisely when $v_{-r} \cdots v_r$ viewed as a single binary number is equal to d_i.

Example. Consider, for example, elementary rule 18 whose local transition rule in tabular form is given by: $(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (0, 1, 0, 0, 1, 0, 0, 0)$. The local transition rule in DNF form is the following:

$$g(v_{-1}, v_0, v_1) = (\neg v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge \neg v_1).$$

A *fuzzy cellular automaton* (FCA) is a particular continuous cellular automaton where the local transition rule is obtained by *DNF-fuzzification* of the local transition rule of a classical Boolean CA. The fuzzification consists of a fuzzy extension of the Boolean operators AND, OR, and NOT in the DNF expression of the Boolean rule. Depending on which fuzzy operators are used, different types of fuzzy cellular automata can be defined. Among the various possible choices, we consider the following: $(a \vee b)$ is replaced by $min\{1, (a+b)\}$, $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1-a)$. Note that, in the case of FCA, $min\{1, (a+b)\} = (a+b)$. Whenever we talk about fuzzification, we are referring to the *DNF-fuzzification* defined above. The resulting local transition rule $f : [0,1]^{2r+1} \to [0,1]$ becomes a real function that generalizes the canonical representation of the corresponding Boolean CA:

$$f(v_{-r}, \cdots, v_r) = \sum_{i<2^{2r+1}} \hat{b}_i \prod_{j=-r:r} l(v_j, d_{i,j+r})$$

where $l(a,0) = 1-a$ and $l(a,1) = a$, and $\hat{b}_i = 0$ if b_i is false and $\hat{b}_i = 1$ if b_i is true. Notice that $\hat{b}_i = g(d_i)$, so we can also write f as:

$$f(v_{-r}, \cdots, v_r) = \sum_{i<2^{2r+1}} g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \tag{1}$$

Note that the resulting function, $f(v_{-r}, \cdots, v_r)$, is affine in each of its variables. Furthermore, it agrees with the Boolean function $g(v_{-r}, \cdots, v_r)$ at the 2^n points in $\{0,1\}^n$. It is therefore the only affine extension of g.

Example. Consider again elementary rule 18 whose local transition rule in DNF form is $g(v_{-1}, v_0, v_1) = (\neg v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge \neg v_1)$, then the corresponding fuzzy local transition rule becomes:

$$f(v_{-1}, v_0, v_1) = (1-v_{-1})(1-v_0)v_1 + v_{-1}(1-v_0)(1-v_1).$$

Throughout this paper, we will denote local rules of Boolean CA by g and their fuzzifications for the corresponding FCA by f. For ease of notation, we will denote $g(y_{i-r}, \cdots, y_i, \cdots, y_{i+r})$ by $g[y_i]$ and $f(x_{i-r}, \cdots, x_i, \cdots, x_{i+r})$ by $f[x_i]$. The corresponding global rules are denoted by G and F.

3 Density Conservation in Boolean and Fuzzy CA

In this section, we begin exploring the link between Boolean and fuzzy CA proving that there are density conservation properties that are preserved through the fuzzification process. Since such properties are defined only for finite or circular CA, throughout this section we will consider circular CA (the finite case is analogous).

3.1 Preliminaries

To begin with, we show that the function obtained through DNF-fuzzification is the only continuous extension of the Boolean function which is affine in every variable. Recall that a function f is affine if it has the form $f(x) = ax + b$ for constants a and b. (An affine function is linear if $b = 0$.) The function $f(x_0, \cdots, x_{n-1})$ is affine in x_0, for example, if it can be written as $a_0(x_1, \cdots, x_{n-1})x_0 + b_0(x_1, \cdots, x_{n-1})$.

Lemma 1. *A local fuzzy rule f obtained from a Boolean rule is affine in each variable.*

Proof. This follows from the construction of f as the sum of terms which are affine in each variable. □

Lemma 2. *The local fuzzy rule f obtained from a Boolean rule is the only continuous extension of the Boolean rule which is affine in each variable.*

Proof. We can think of a function $f(x_0, \cdots, x_{n-1})$ which is affine in each variable x_i as nested affine equations in each variable:

$$f(x_0, \cdots, x_{n-1}) = a_{n-1}(\cdots(a_1(a_0x_0 + b_0)x_1 + b_1)\cdots)x_{n-1} + b_{n-1}.$$

Since there are 2^n parameters, a_i and b_i, such an equation is completely defined by 2^n points. As an extension of the Boolean, the function f is defined on the 2^n points in $\{0, 1\}^n$, and is thus the unique affine extension. □

The lemmas above imply that the global rule F obtained from such local rules are affine in each variable at each position.

3.2 Number Conservation

Number conservation is a global property that has been extensively investigated (e.g., see [5, 6, 11, 12, 15, 23, 24]) since its introduction in [25], a main focus being the study of linear time decision algorithms for the property of number conservation for finite or periodic configurations.

A Boolean CA is number conserving if the number of ones in the initial configuration is preserved at each subsequent iteration (we will also say that a rule is number conserving). The analogous property in fuzzy CA is that the sum of values of the initial configuration is preserved.

In this section, we wish to show that using DNF-fuzzification, a Boolean CA with local rule g is number conserving if and only if the local rule f of the corresponding FCA is sum conserving (Theorem 3). We will actually first prove a more general result that holds for any linear function (Theorem 2). Before starting the proofs of the lemmas leading to the main theorems, we introduce an extension of the fuzzy rule, and some notation necessary for the proofs.

Let us extend the fuzzification process to any function $g : \{0, 1\}^{2r+1} \to \mathbb{R}$ by defining

$$C(g)(v_{-r}, \cdots, v_r) = \sum_{i<2^{2r+1}} g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r})$$

where d_i and $l(v_j, d_{i,j+r})$ are defined as for the fuzzification of g. The function $C(g)$: $\mathbb{R}^{2r+1} \to \mathbb{R}$ is once again affine in each variable. Similarly, we can define $C(G)$ to be the continuous extension of any global function G. Notice that when g is a local Boolean rule, $C(g) = f$.

Let $\sigma(x_0, \cdots, x_{n-1})$ denote $\sum_{i=0}^{n-1} x_i$. Then any linear function $\Psi = \sum_{i=0}^{n-1} \alpha_i x_i$ can be written as the composition of σ and a scaling function, that is, a function of the form $\psi(x_0, \cdots, x_{n-1}) = (\alpha_0 x_0, \cdots, \alpha_{n-1} x_{n-1})$ where the α_i are constants. Let us further define $C(\psi \circ G)(x_0, \cdots, x_{n-1})$ to be $(C(\alpha_0 g)[x_0], \cdots, C(\alpha_{n-1} g)[x_{n-1}])$. We will now prove a few lemmas regarding the composition of $C(g)$ with linear functions that will be needed for the proof of the main Theorem.

Lemma 3. *Given any linear function $\Psi = \sigma \circ \psi$ and a local function g with associated global function G, then $\Psi \circ C(G) = \sigma \circ C(\psi \circ G)$ and is affine in each variable.*

Proof. We prove the slightly stronger statement, $\psi \circ C(G) = C(\psi \circ G)$ by first showing that $\alpha C(g) = C(\alpha g)$:

$$\begin{aligned} C(\alpha g)(v_{-r}, \cdots, v_r) &= \sum_{i<2^{2r+1}} \alpha g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \\ &= \alpha \sum_{i<2^{2r+1}} g(d_i) \prod_{j=-r:r} l(v_j, d_{i,j+r}) \\ &= \alpha C(g). \end{aligned}$$

Then

$$\begin{aligned} C(\psi \circ G)(x_0, \cdots, x_{n-1}) &= (C(\alpha_0 g)[x_0], \cdots, C(\alpha_{n-1} g)[x_{n-1}]) \\ &= (\alpha_0 C(g)[x_0], \cdots, \alpha_{n-1} C(g)[x_{n-1}]) \\ &= \psi \circ C(G)(x_0, \cdots, x_{n-1}). \end{aligned}$$

Hence $\Psi \circ C(G) = \sigma \circ \psi \circ C(G) = \sigma \circ C(\psi \circ G)$.

As a sum of functions which are affine in each variable, $\Psi \circ C(G)$ is affine in each variable also. □

Lemma 4. *Let $\Psi = \sigma \circ \psi$ and $\Phi = \sigma \circ \phi$ be linear functions and let G be a global function such that $\Psi \circ G = \tilde{\Phi}$ on $\{0,1\}^n$, then $\sigma \circ C(\psi \circ G) = \sigma \circ C(\phi)$ on $\mathbb{R}^n$.*

Proof. We prove this by induction on the number of entries in the configuration that can be in $(0, 1)$, all other variables being in $\{0, 1\}$. The theorem is clearly true when all x_i are in $\{0, 1\}$. Now assume it is true when m variables x_i can range over $[0, 1]$, the rest being strictly in $\{0, 1\}$ and prove for $m + 1$. For ease of notation and without loss of generality let x_0 be allowed to range over $[0, 1]$. Then from Lemma 3, $\sigma \circ C(\psi \circ G)$ and $\sigma \circ C(\phi)$ are affine as functions in x_0. By the induction hypothesis, these affine functions must agree with Φ when x_0 is equal to 0 or 1. Since two points uniquely determine an affine function, we must have $\sigma \circ C(\psi \circ G) = \sigma \circ C(\phi)$. □

We are now ready to prove our main results.

Theorem 1. *Given linear functions* Ψ *and* Φ *from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *and any function* $g : \{0,1\}^{2r+1} \to \mathbb{R}$*, then*

$$\Psi \circ C(G) = \Phi$$

if and only if

$$\Psi \circ G = \tilde{\Phi}$$

where $\tilde{\Phi}$ *is the restriction of* Φ *to* $\{0,1\}^n$*, and* G *is the global function associated with* g*.*

Proof. $\Rightarrow$: Since the property applies to all values in $[0,1]$, it must apply to $\{0,1\}$ as well and the implication follows from the construction of f.
$\Leftarrow$:

$$\begin{aligned} \Psi \circ G &= \sigma \circ C(\psi \circ G) \text{ by Lemma 3} \\ &= \sigma \circ C(\tilde{\phi}) \text{ by Lemma 4} \\ &= \Phi \text{ by Lemma 2 since } C(\tilde{\phi}) \text{ is affine.} \end{aligned}$$

□

Theorem 2. *Let* Ψ *be a real linear function and* g *a local Boolean CA rule. Then:*

$$\forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n \quad \Psi(g[y_0], \cdots, g[y_{n-1}]) = \Psi(y_0, \cdots, y_{n-1})$$

if and only if

$$\forall (x_0, \ldots, x_{n-1}) \in [0,1]^n \quad \Psi(f[x_0], \cdots, f[x_{n-1}]) = \Psi(x_0, \cdots, x_{n-1})$$

Proof. This follows from Theorem 1 by letting $\Psi = \Phi$ and the fact that $f = C(g)$.

□

Note that, when Ψ is the summation of all values, we have: $\sum_{i=0}^{n-1} g[y_i] = \sum_{i=0}^{n-1} y_i$ $\forall (y_0, \ldots, y_{n-1})$ if and only if $\sum_{i=0}^{n-1} f[x_i] = \sum_{i=0}^{n-1} x_i$ $\forall (x_0, \ldots, x_{n-1})$, that is:

Theorem 3. *A Boolean CA is number conserving if and only if its corresponding FCA is sum conserving.*

Example: Rule 184 is an example of a number conserving rule.

Theorem 4. *Let* f_{184} *be fuzzy local rule 184. We have:*

$$\forall (x_0, \ldots, x_n) \in [0,1]^n \quad \sum_{i=0}^{n-1} f_{184}[x_i] = \sum_{i=0}^{n-1} x_i$$

Proof. Fuzzy rule 184 has the following form: $x_i^{t+1}x_{i-1}^t - x_{i-1}^t x_i^t + x_i^t x_{i+1}^t$. Then we have:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_{i-1}^t - \sum_{i=0}^{n-1} x_i^t x_{i-1}^t + \sum_{i=0}^{n-1} x_i^t x_{i+1}^t.$$

Since we are using a circular FCA, $\sum_{i=0}^{n-1} x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t$ and $\sum_{i=0}^{n-1} x_i^t x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t x_{i+1}^t$, which implies:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_i^t.$$

□

The result for the Boolean case (which is already known) follows as a corollary of Theorem 2.

Corollary 1. *Let g_{184} be elementary Boolean local rule 184. We have:*
$\forall (y_0, \ldots, y_n) \in \{0,1\}^n \;\; \sum_{i=0}^{n-1} g_{184}[y_i] = \sum_{i=0}^{n-1} y_i$.

3.3 Spatial Number Conservation

We now describe another global property that is preserved by fuzzification. This property also deals with the density of configurations. Following an approach similar to the one of Theorem 2, we can show that in a CA, linear properties hold for the Boolean rule if and only if they hold for the corresponding fuzzy rule.

Theorem 5. *Let $g : \{0,1\}^{2r+1} \to \{0,1\}$ be the local rule of a Boolean CA and let $f : [0,1]^{2r+1} \to [0,1]$ be its fuzzification. Let Ψ be a real linear function.*

$$\forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n \;\; \Psi(g[y_0], \cdots, g[y_{n-1}]) = 0$$

if and only if

$$\forall (x_0, \ldots, x_{n-1}) \in [0,1]^n \;\; \Psi(f[x_0], \cdots, f[x_{n-1}]) = 0.$$

Proof. This follows from Theorem 1 by letting $\Phi = 0$. □

Note that, when $\Psi(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} (-1)^i x_i$ and n is even, we obtain the preservation through fuzzyfication of a spatial conservation property where the sum of the even numbered cells (x_{2i}) is equal to the sum of the odd numbered cells (x_{2i+1}) at any time after the initial configuration:

Corollary 2. *Let n be even.* $\forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n \;\; \sum_{i=0}^{n-1} (-1)^i g[y_i] = 0$
if and only if
$\forall (x_0, \ldots, x_{n-1}) \in [0,1]^n \;\; \sum_{i=0}^{n-1} (-1)^i f[x_i] = 0$.

Example: Rule 46 is an example of a spatially number conserving rule where the sum of the even numbered cells (x_{2i}) is equal to the sum of the odd numbered cells (x_{2i+1}) at any time after the initial configuration.

Theorem 6. *Let f_{46} be fuzzy local rule 46 in a FCA of even size. We have:* $\forall(x_0,\ldots,x_{n-1}) \in [0,1]^n \;\; \sum_{i=0}^{n-1}(-1)^i f_{46}[x_i] = 0$.

Proof. Rule 46 is given by: $x_i^{t+1} = x_i^t + x_{i+1}^t - x_{i-1}^t x_i^t - x_i^t x_{i+1}^t$, so:

$$\sum_{i=0}^{n-1}(-1)^i x_i^{t+1} = \sum_{i=0}^{n-1}(-1)^i x_i^t + \sum_{i=0}^{n-1}(-1)^i x_{i+1}^t - \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t - \sum_{i=0}^{n-1}(-1)^i x_i^t x_{i+1}^t.$$

By a change of variables, due to circularity we have: $\sum_{i=0}^{n-1}(-1)^i x_{i+1}^t = -(\sum_{i=0}^{n-1}(-1)^i x_i^t)$, and $\sum_{i=0}^{n-1}(-1)^i x_i^t x_{i+1}^t = -(\sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t)$. So we can conclude:

$$\sum_{i=0}^{n-1}(-1)^i x_i^{t+1} = \sum_{i=0}^{n-1}(-1)^i x_i^t - \sum_{i=0}^{n-1}(-1)^i x_i^t - \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t + \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t = 0.$$

□

The result for the Boolean case now follows as a corollary of Theorem 2.

Corollary 3. *Let g_{46} be elementary Boolean local rule 46. When n is even, we have:* $\forall(y_0,\ldots,y_{n-1}) \in \{0,1\}^n \;\; \sum_{i=0}^{n-1}(-1)^i g_{46}[y_i] = 0$.

4 Self-Oscillation and Additivity

In this section, we consider another property of Boolean cellular automata extensively studied in the literature: additivity (e.g., see [8, 18, 28]). We continue the investigation of the link between Boolean and fuzzy CA showing a connection between additivity and a new fuzzy property that we call *self-oscillation*. In doing so, we characterize the class of self-oscillating fuzzy CA. Although for simplicity we take our examples from one dimensional CA, the results of this section hold for any dimension.

4.1 Preliminaries

A common definition of additivity is that a Boolean rule g is *additive* if $g(y_0,\cdots,y_{n-1}) \oplus g(z_0,\cdots,z_{n-1}) = g(y_0 \oplus z_0\ ,\cdots,y_{n-1} \oplus z_{n-1})$. These additive rules can be expressed as the XOR of some of their variables. An example is elementary rule 90, which can be expressed as: $g_{90}(x,y,z) = (\bar{x} \wedge z) \vee (x \wedge \bar{z}) = x \oplus z$. We will use a broader definition of additivity which includes rules that are additive by the definition above and their negations:

Definition 1. *A Boolean rule g is* additive *if*

$$g(y_0, \cdots, y_{n-1}) \oplus g(z_0, \cdots, z_{n-1}) = g(y_0 \oplus z_0, \cdots, y_{n-1} \oplus z_{n-1}),$$

or

$$g(y_0, \cdots, y_{n-1}) \oplus g(z_0, \cdots, z_{n-1}) = \overline{g(y_0 \oplus z_0, \cdots, y_{n-1} \oplus z_{n-1})}.$$

An example of a rule which is additive in this broader sense but not by the strict mathematical definition is rule $g_{105}(x, y, z) = xy\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} = \overline{x \oplus y \oplus z}$, which is equal to $x \oplus y \oplus \bar{z} = x \oplus \bar{y} \oplus z = \bar{x} \oplus y \oplus z$.

$$\begin{aligned} g_{105}(x_1, y_1, z_1) \oplus g_{105}(x_2, y_2, z_2) &= x_1 \oplus y_1 \oplus \bar{z}_1 \oplus x_2 \oplus y_2 \oplus \bar{z}_2 \\ &= x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus z_1 \oplus z_2 \end{aligned}$$

while

$$\begin{aligned} g_{105}(x_1 \oplus x_2, y_1 \oplus y_2, z_1 \oplus z_2) &= x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus \overline{z_1 \oplus z_2} \\ &= \overline{x_1 \oplus x_2 \oplus y_1 \oplus y_2 \oplus z_1 \oplus z_2}. \end{aligned}$$

In general, when g is additive, $g(y_0, \cdots, y_{n-1})$ can be expressed as the XOR of some of its variables y_i and at most one negation $\bar{y}_i$, which implies the following property:

Property 1. *An additive Boolean rule has the form:* $g(x_0, \cdots, x_{n-1}) = \bigoplus_{i \in S} x_i$ *or* $g(x_0, \cdots, x_{n-1}) = \overline{\bigoplus_{i \in S} x_i}$*, where i ranges over S, a subset of the numbers from 0 to $n-1$.*

We extend the definition of the XOR operator to fuzzy rules by defining $x \oplus y = x\bar{y} + \bar{x}y = x(1-y) + (1-x)y$. (In this section, to simplify notation we will often use $\bar{x}$ to denote $(1-x)$ in a fuzzy rule.) If a Boolean rule is additive, its fuzzification is also additive and Property 1 holds for fuzzy rules as well.

A *fixed point* $\mathbf{P}$ for a FCA with global transition rule F is a configuration $\mathbf{P}$ such that $F(\mathbf{P}) = \mathbf{P}$. A configuration $\mathbf{P} = (\ldots, p_{i-1}, p_i, p_{i+1}, \ldots)$ is *homogeneous* if $p_i = p_j, \forall i, j$; in such a case, we obviously also have $f(p, \cdots, p) = p$. A global rule is said to *converge* to an homogeneous configuration $\mathbf{P} = (\ldots p, p, p, \ldots)$ if, for all initial configurations $\mathbf{X^0} = (\ldots, x^0_{i-1}, x^0_i, x^0_{i+1} \ldots)$ with $x^0_i \in (0, 1)$ for all i, then $\forall \epsilon > 0$, $\exists T$ such that $\forall t > T$ and $\forall i$, $|x^t_i - p| < \epsilon$. In this case, we will also say that the local rule f *converges to* p. Note that if a rule converges to a homogeneous configuration it must be a fixed point.

We can now introduce the notion of self-oscillation for fuzzy CA. Informally, a fuzzy rule f is *s*elf-oscillating if while converging towards an homogeneous fixed point, it behaves like the corresponding Boolean rule g; in other words, when the dynamics of f around a fixed point coincides with the dynamics of g. In fact, the rule table of a fuzzy self-oscillating CA, written around its fixed point, coincides with the Boolean rule table. This is the case, for example, of elementary fuzzy rule 90 which has been shown in [14] to

x	y	z	$f_{90}(x,y,z)$		x	y	z	$g_{90}(x,y,z)$
$<$	$<$	$<$	$<$		0	0	0	0
$<$	$<$	$>$	$>$		0	0	1	1
$<$	$>$	$<$	$<$		0	1	0	0
$<$	$>$	$>$	$>$		0	1	1	1
$>$	$<$	$<$	$>$		1	0	0	1
$>$	$<$	$>$	$<$		1	0	1	0
$>$	$>$	$<$	$>$		1	1	0	1
$>$	$>$	$>$	$<$		1	1	1	0

Table 1: Rule 90: fuzzy behaviour around $\frac{1}{2}$ (< indicates "<$\frac{1}{2}$", > indicates ">$\frac{1}{2}$") (left); Boolean rule (right).

behave like its Boolean counter-part around $\frac{1}{2}$. (See Table 1 where $>$ and $<$ respectively indicate values greater than or smaller than $\frac{1}{2}$.)

We now introduce the formal definition of self-oscillation. Let p be a fixed point for f. Let $(x_1, \ldots, x_{n-1})$ be an arbitrary fuzzy configuration, let $x_n = f(x_0, \cdots, x_{n-1})$, and let us define y_i, for $i = 0, \ldots n$, as follows:

$$y_i = \begin{cases} 0 & \text{if } x_i < p \\ 1 & \text{if } x_i > p \end{cases}$$

Definition 2. *Rule f is* self-oscillating *around p if it converges to p and if $f(x_0, \cdots, x_{n-1}) = x_n$ implies that $g(y_0, \cdots, y_{n-1}) = y_n$.*

Elementary rule 90 has been shown to have this type of behaviour in [14]. The other self-oscillating elementary rules have been identified using a case by case analysis in [2]. However, the general implications of this behaviour were left unexplained. What was clear was that self-oscillation did not occur for all fuzzy rules with an homogeneous fixed point, but a characterization of the class of rules displaying self-oscillation was lacking until now.

4.2 Equivalence between Self-Oscillation and Additivity

In this section, we characterize the class of self-oscillating FCA proving the following result: *a non-trivial fuzzy CA rule is self-oscillating if and only if the corresponding Boolean CA rule is additive*.

We begin with some lemmas. We first describe the behaviour of the fuzzification of the XOR operator ($x \oplus y = x\bar{y} + \bar{x}y$) around $\frac{1}{2}$, and then prove that convergence to $\frac{1}{2}$ is necessary for self-oscillation.

Lemma 5. *$xy + \bar{x}\bar{y}$ is greater than $\frac{1}{2}$ if and only if both x and y are greater than $\frac{1}{2}$ or both are smaller.*

Lemma 6. *A necessary condition for a convergent non-trivial rule to be self-oscillating is for it to converge to one half.*

Proof. To begin we note that functions converging to either zero or one can never be self-oscillating since values are, respectively, either always greater than or always less than the point of convergence. We will now prove this lemma by induction on n, the number of variables in f, i.e., on the size of the neighbourhood.

It is easy (but tedious) to show that when f is a non-trivial function on two variables only the following converge to homogeneous fixed points on $(0, 1)$: $f_1(x_0, x_1) = x_0\bar{x}_1 + \bar{x}_0x_1$ and $f_2(x_0, x_1) = x_0x_1 + \bar{x}_0\bar{x}_1$ which converge to $\frac{1}{2}$ and are self-oscillating, and $f_3(x_0, x_1) = \bar{x}_0\bar{x}_1$ which converges to $p = \frac{3-\sqrt{5}}{2}$ and is not self-oscillating. For f_3 to be self-oscillating, it would have to be greater than p only when both x_0 and x_1 were less than p. A counterexample occurs when $x_0 = 0$ and $x_1 = \frac{1}{2}$, then $f_3(x_0, x_1) = \frac{1}{2} > p$.

Now assume that the lemma holds for all functions in n or fewer variables and consider the function f with global rule F which converges to a fixed point p. We re-write it as: $f_+(x_0, \cdots, x_{n-1})x_n + f_-(x_0, \cdots, x_{n-1})\bar{x}_n$. We wish to show that if f is convergent and non-trivial, then at least one of f_+ and f_- must take on values greater than and less than p. If both f_+ and f_- are always greater than p, then $f > px_n + p(1 - x_n) = p$. Self-oscillation implies that $f = 1$. Similarly, if f_+ and f_- are both less than p, then f must be trivially 0. Now consider f_+ always greater than p and f_- always less than p. When $x_n = 1$, $f(x_0, \cdots, x_{n-1}, 1) = f_+(x_0, \cdots, x_{n-1}) > p$. Self-oscillation implies that $f(x_0, \cdots, x_n) > p$ whenever $x_n > p$. When $x_n = 0$, $f(x_0, \cdots, x_{n-1}, 0) = f_-(x_0, \cdots, x_{n-1}) < p$. Again, self-oscillation implies $f < p$ whenever $x_n < p$. Taking the two together, we must have $f(x_0, \cdots, x_n) = x_n$ which is not a convergent function. Similarly, if $f_+ < p$ and $f_- > p$, we obtain $f = \bar{x}_n$. We conclude that at least one of f_+ and f_- must have some values greater than p and some smaller. Assume, without loss of generality since the proofs are analogous, that f_+ is sometimes greater than p and sometimes smaller, and again consider $x_n = 1$ so that $f(x_0, \cdots, x_{n-1}, 1) = f_+(x_0, \cdots, x_{n-1})$. The function f_+ is completely determined by f and so must be self-oscillating around p. By the inductive hypothesis, $p = \frac{1}{2}$. □

As we know, given a Boolean rule g, we can derive its fuzzification f as the sum of the fuzzifications of each of its transitions to 1. In the following, we refer to each of the products in this sum as a *term of* f.

Lemma 7. *If* $f(x_0, \cdots, x_{n-1})$ *converges to* $\frac{1}{2}$, f *is the sum of* 2^{n-1} *terms.*

Proof. The terms of any function evaluated at $(\frac{1}{2}, \cdots, \frac{1}{2})$ are all equal to $(\frac{1}{2})^n$. For $f(\frac{1}{2}, \cdots, \frac{1}{2}) = \frac{1}{2}$, we must have 2^{n-1} such terms summed together. □

We now prove that for a fuzzy rule on n variables to be self-oscillating, it must be balanced in x_i and $\bar{x}_i$. That is, it must be the sum of the same number of terms in x_i as in $\bar{x}_i$ for all i.

Lemma 8. *Let $f(x_0, \cdots, x_{n-1})$ be self-oscillating. Then for all i, there are as many terms in the sum of f in x_i as there are terms in $\bar{x}_i$.*

Proof. We will show by contradiction that there are as many terms in the sum of f in x_i as there are terms in $\bar{x}_i$. We begin by writing f as:

$$f(x_0, \cdots, x_{n-1}) =$$

$$f_{i+}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1})x_i + f_{i-}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1})\bar{x}_i$$

Assume without loss of generality that more than half the terms are in f_+. Let there be $m > 2^{n-2}$ terms in f_{i+}. Then, by Lemma 7 there must be $2^{n-1} - m$ terms in f_{i-}. Then as $x_j \to \frac{1}{2}$ for all $j \neq i$, each term of f_{i+} tends to $\frac{1}{2}^{n-1}$ and thus $f_{i+} \to \frac{m}{2^{n-1}}$, which is $> \frac{1}{2}$ because we assumed $m > 2^{n-2}$. Moreover, $f_{i-} \to \frac{2^{n-1}-m}{2^{n-1}} < \frac{1}{2}$. Note that this convergence happens as the x_j approach $\frac{1}{2}$ from both directions. Choosing x_j close enough to $\frac{1}{2}$, we can assume that $f_{i+} > \frac{1}{2}$ and $f_{i-} < \frac{1}{2}$. Now: $f(x_0, \cdots, x_{n-1}) = f_{i+}x_i + f_{i-}(1 - x_i) = (f_{i+} - f_{i-})x_i + f_{i-}$. At $x_i = 1$, $f(x_0, \cdots, x_{n-1}) = f_{i+} > \frac{1}{2}$. That is for all values of $x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1}$ close enough to $\frac{1}{2}$, whether greater than or less than $\frac{1}{2}$, $f(x_0, \cdots, x_{n-1}) = f_{i+} > \frac{1}{2}$. Similarly, when $x_i = 0$, $f(x_0, \cdots, x_{n-1}) = f_{i+} < \frac{1}{2}$. Self-oscillation then implies that $f(x_0, \cdots, x_{n-1}) = x_i$ which is not a convergent function. □

We are finally able to characterize the form of a self-oscillating rule. We will see that these rules are fuzzifications of Boolean rules which are the XOR of single variables or their negations.

Theorem 7. *A rule $f(x_0, \cdots, x_{n-1})$ is self-oscillating if and only if its corresponding Boolean rule is additive.*

Proof. $\Rightarrow$:
We will prove that if a self-oscillating rule is additive, $f(x_0, \cdots, x_{n-1}) = \bigoplus_{i \in S} x_i$ or $f(x_0, \cdots, x_{n-1}) = \overline{\bigoplus_{i \in S} x_i}$, (and thus the corresponding Boolean rule is additive) by induction on n.

For $n = 2$, from Lemma 8, we must have one term in x_i and one term in $\bar{x}_i$ for $i \in \{0, 1\}$ giving us only two possibilities: $f(x_0, x_1) = x_0\bar{x}_1 + \bar{x}_0 x_1 = x_0 \oplus x_1$ or $f(x_0, x_1) = \bar{x}_0\bar{x}_1 + x_0 x_1 = \bar{x}_0 \oplus x_1$ as required.

Now assume the hypothesis for all self-oscillating rules in less than or equal to n variables. Given a self-oscillating rule $f(x_0, \cdots, x_n)$, if f is not dependent on all $n + 1$ variables, then it can be rewritten as a self-oscillating rule on n or fewer variables and the inductive hypothesis holds. So we may continue on the assumption that f is dependent on all $n + 1$ variables. We can write:

$$\begin{aligned} f(x_0, \cdots, x_n) &= [f_{1-}(x_0, \cdots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \cdots, x_{n-2})x_{n-1}]\bar{x}_n \\ &+ [f_{2-}(x_0, \cdots, x_{n-2})\bar{x}_{n-1} + f_{2+}(x_0, \cdots, x_{n-2})x_{n-1}]x_n \end{aligned}$$

Letting $x_n = 0$, $f(x_0, \cdots, x_{n-1}, 0)$ is a self-oscillating rule on n variables so the inductive hypothesis applies and

$$f_{1-}(x_0, \cdots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \cdots, x_{n-2})x_{n-1} = x_0 \oplus \cdots \oplus x_{n-1}$$

or

$$f_{1-}(x_0, \cdots, x_{n-2})\bar{x}_{n-1} + f_{1+}(x_0, \cdots, x_{n-2})x_{n-1} = \bar{x}_0 \oplus x_1 \oplus \cdots \oplus x_{n-1}.$$

Specifically, we must have $f_{1-}(x_0, \cdots, x_{n-2}) = x_0 \oplus x_1 \oplus \cdots \oplus x_{n-2}$, $f_{1+}(x_0, \cdots, x_{n-2}) = \bar{x}_0 \oplus x_1 \oplus \cdots \oplus x_{n-2}$ or the opposite. Setting x_n to 1, we can say the same thing about f_{2-} and f_{2+}.

Using the same argument, if we let $x_{n-1} = 0$, we see that $f_{2-} = \bar{f}_{1-}$. Thus we have only two possibilities for f:

$$\begin{aligned} f(x_0, \cdots, x_n) &= [(x_0 \oplus \cdots \oplus x_{n-2})\bar{x}_{n-1} + (\bar{x}_0 \oplus \cdots \oplus x_{n-2})x_{n-1}]\bar{x}_n \\ &+ [(\bar{x}_0 \oplus \cdots \oplus x_{n-2})\bar{x}_{n-1} + (x_0 \oplus \cdots \oplus x_{n-2})x_{n-1}]x_n \\ &= x_0 \oplus \cdots \oplus x_n \end{aligned}$$

or

$$\begin{aligned} f(x_0, \cdots, x_n) &= [(\bar{x}_0 \oplus \cdots \oplus x_{n-2})\bar{x}_{n-1} + (x_0 \oplus \cdots \oplus x_{n-2})x_{n-1}]\bar{x}_n \\ &+ [(x_0 \oplus \cdots \oplus x_{n-2})\bar{x}_{n-1} + (\bar{x}_0 \oplus \cdots \oplus x_{n-2})x_{n-1}]x_n \\ &= \bar{x}_0 \oplus x_1 \oplus \cdots \oplus x_n. \end{aligned}$$

$\Leftarrow$: We will assume that the Boolean rule corresponding to f is additive (and thus $f(x_0, \cdots, x_{n-1})$ is also additive) and proceed by induction on n to show that it is self-oscillating. When $n = 2$, $f(x_0, x_1)$ is equal to $x_0 \oplus x_1$ or $x_0 \oplus \bar{x}_1$. In either case, by Lemma 5 f is self-oscillating.

Now assume that for n or fewer variables, additivity implies self-oscillation and consider $f(x_0, \cdots, x_n)$. Without loss of generality, assume that f is not independent of x_n, then we can write it as $f(x_0, \cdots, x_n) = f_1(x_0, \cdots, x_{n-1}) \oplus x_n$ for an additive rule f_1 which is self-oscillating by the induction hypothesis. Again applying Lemma 5, f must be self-oscillating. □

If we restrict our examination to elementary rules, we obtain the following corollary.

Corollary 4. *Up to equivalence, all and only rules f_{60}, f_{90}, f_{105}, and f_{150} are elementary self-oscillating rules.*

5 Probabilistic Interpretation of Fuzzification

An interesting property of the DNF fuzzification is how it relates to the probability of a one occuring at a given time in a given cell. Since the fuzzy values are in the range $[0, 1]$, we can interpret them as probabilities, i.e., we can let a fuzzy value x_i^t denote the probability that a cell y_i of a Boolean CA assumes value 1 at time t. Then, if the values were independent, the fuzzy rule applied to a neighbourhood would return the probability of having value 1 at the next time step:

$$f(x_{i-r}^t, \cdots, x_i^t, \cdots, x_{i+r}^t) = x_i^{t+1} = P(y_i^{t+1} = 1).$$

In the next section we will establish some basic probabilistic results resulting from this interpretation.

5.1 Preliminaries

We introduce a property that will be needed later, relating the expectation of a Boolean local function to the fuzzy rule applied to expectations.

We will first need some notation. Given a random variable Z, let $E(Z)$ denote its expected value. Note that when Z is a binary random variable, then $E(Z)$ is the probability $P(Z = 1)$. Essentially, we show that applying the fuzzification f of g to the expected values of a cell Y_i and its $2r$ neighbouring cells, we obtain the expected value of $g[Y_i]$, the cell at the next time step.

Theorem 8. *Let* $(Y_0, \ldots, Y_{n-1})$ *be independent binary random variables. Then:* $\forall i = 0, \ldots n-1, \quad f[E(Y_i)] = E(g[Y_i])$.

Proof. By definition, $f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \prod_{k=-r}^{r} l(E(Y_{i+k}), d_{j,k+r})$.

If $d_{j,k+r} = 1$, then

$$l(E(Y_{i+k}), d_{j,k+r}) = E(Y_{i+k}) = P(Y_{i+k} = d_{j,k+r}).$$

Similarly, if $d_{j,k+r} = 0$, then

$$l(E(Y_{i+k}), d_{j,k+r}) = 1 - E(Y_{i+k}) = 1 - P(Y_{i+k} = 1) = P(Y_{i+k} = 0) = P(Y_{i+k} = d_{j,k+r}).$$

So we have:

$$f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \prod_{k=-r}^{+r} P(Y_{i+k} = d_{j,k+r})$$

Since the variables are independent,

$$\prod_{k=-r}^{+r} P(Y_{i+k} = d_{j,k+r}) = P((Y_{i-r}, \ldots, Y_{i+r}) = d_j)$$

thus:

$$f[E(Y_i)] = \sum_{j=0}^{2^{2r+1}-1} b_j \cdot P((Y_{i-r}, \ldots, Y_{i+r}) = d_j).$$

Recall that $b_j = 1$ if $d_j \in \mathcal{T}_1$, the set of Boolean tuples mapping to one, otherwise $b_j = 0$, thus:

$$\begin{aligned} f[E(Y_i)] &= \sum_{d_j \in \mathcal{T}_1} P((Y_{i-r}, \ldots, Y_{i+r}) = d_j) \\ &= P((Y_{i-r}, \cdots, Y_{i+r}) \in \mathcal{T}_1) \\ &= P(g[Y_i] = 1) \\ &= E(g[Y_i]). \end{aligned}$$

□

As a consequence of Theorem 8, we can intuitively see that the asymptotic behaviour of a FCA represents a rough approximation of the asymptotic density of the corresponding Boolean CA. In the next section, we show that such an intuition is in fact correct.

5.2 Mean Field Approximation

In this section, we will show the connection between the asymptotic behaviour of fuzzy CA and of one descriptor of the asymptotic behaviour of Boolean CA.

The mean field approximation is an estimate of the asymptotic density of Boolean cellular automata when no spatial correlation among cells is taken into account. Thought of another way, it is again an estimate of the probability of a one occurring in a random place in a configuration once its density has stabilized [16, 30], not considering spatial correlations. Although in cellular automata spatial correlations play an important role and greatly influence their dynamics, the mean field approximation can give a rough indication, although sometimes quite far from the exact value, of the asymptotic density. The approximation is derived by assuming that when the asymptotic probability is reached, then the likelihood of increasing in density is equal to the likelihood of decreasing in density. More formally, we assume that for all i, $P(y_i = 1) = p$ and that the y_i are independent. Then we can denote the probability of a transition from 0 to 1 as a function of p by $P_{0\to1}(p)$. This is equal to the probability that $g[y_i] = 1$ given that $y_i = 0$ or $P(g[y_i] = 1|y_i = 0)$. Similarly, we denote the probability of a transition from 1 to 0 by $P_{1\to0}(p)$. A stable density of the mean field approximation is any p such that $P_{0\to1}(p) - P_{1\to0}(p) = 0$. We show in the following lemma that these probabilities can be evaluated as the sum of fuzzifications of the transitions from 0 to 1 evaluated at p which we denote by $R_{0\to1}(p)$, in the first instance, and as $R_{1\to0}(p)$ the sum of fuzzifications of the transitions from 1 to 0 also evaluated at p, in the second.

Lemma 9. *$P_{0\to1}(p) = R_{0\to1}(p)$ and $P_{1\to0}(p) = R_{1\to0}(p)$.*

Proof. We prove that $P_{0\to1}(p) = R_{0\to1}(p)$, the analogous proof holds for $P_{1\to0}(p) = R_{1\to0}(p)$. First note that since in the calculation of the mean field approximation we are assuming that the y_i are independent, the probability of any given neighbourhood combination $[y_i]$ is the fuzzification of that neighbourhood evaluated at p. That is, let $(v_{-r}, \cdots, v_r)$ be a binary vector, then $P((y_{i-r}, \cdots, y_{i+r}) = (v_{-r}, \cdots, v_r)) = \prod_{j=-r:r} l(p, v_j)$ where as before $l(p,1) = p$ and $l(p,0) = 1-p$. By definition, $P_{0\to 1}(p)$ is the probability that $g[y_i] \in \tau_1$ given that $y_i = 0$, so it is equal to the sum of the fuzzifications of the transitions from 0 to 1, or $R_{0\to 1}(p)$. □

Theorem 9. *Given a global fuzzy rule F, if there exists an homogeneous configuration $X = (p, \cdots, p)$ such that $F(X) = X$, then p is a stable density of the mean field approximation of the Boolean rule G associated with F.*

Proof. Let f be the local rule associated with F and g its Boolean rule. Let $R_{0\to 1}(p)$ denote the sum of the fuzzifications of the transitions from 0 to 1 for g, evaluated at $X = (p, \cdots, p)$. Similarly, $R_{0\to 0}(p)$, $R_{1\to 0}(p)$, and $R_{1\to 1}(p)$ denote sums of fuzzifications of transitions from 0 to 0, 1 to 0, and 1 to 1 evaluated at $(p, \cdots, p)$, respectively. The sum of all these transition must be one. Since X is fixed by F, and since $f(p, \cdots, p) = R_{0\to 1}(p) + R_{1\to 1}(p)$ by definition, then $R_{0\to 1}(p) + R_{1\to 1}(p) = p$. Also, $R_{1\to 0}(p) + R_{1\to 1}(p) = p$ since this is the sum of all terms in x_i (as opposed to terms in $\bar{x}_i$), and the result is independent of f. Combining these two results, we have

$$\begin{aligned} (R_{1\to 0}(p) + R_{1\to 1}(p)) - (R_{0\to 1}(p) + R_{1\to 1}(p)) &= p - p \\ R_{1\to 0}(p) - R_{0\to 1}(p) &= 0. \end{aligned}$$

Thus at p, $P_{1\to 0}(p) = P_{0\to 1}(p)$ by Lemma 9 . Hence p is a stable density of the mean field approximation, as required.

Note that if p is not unique, then the mean field approximation has several stable densities. □

It is easy to see that the reverse also holds.

Theorem 10. *If p is a stable density of the mean field approximation for a Boolean rule G, then the homogeneous configuration at that point is a fixed point for the fuzzification F of G.*

In the next section, we show further connections between the density of Boolean CA and their corresponding fuzzy CA.

6 Concluding Remarks

In this paper, we have provided the first evidence of a link between Boolean and fuzzy cellular automata by focusing on density conservation and additivity. We have formally proven that density conservation is preserved through fuzzification and that additivity in Boolean CA is equivalent to self-oscillation in FCA.

Now that there is a formal proof of strong links between the discrete and the continuous models, the next natural question is how to exploit these links to derive properties of Boolean cellular automata through their fuzzification. As a consequence of our results, we have started the investigation in this direction showing that density conservation in Boolean CA could indeed be easily derived from fuzzy sum preservation and, in addition, we have uncovered a spatial density conservation in Boolean CA through the study of the continuous version. Furthermore, we have shown a link between additivity in Boolean CA and the asymptotic behaviour of fuzzy CA. An interesting research direction would be to examine the link between surjectivity and injectivity in Boolean CA and the asymptotic behaviour of fuzzy CA.

Finally, the link between DNF fuzzification and mean field approximation opens intriguing research directions: when a fuzzy CA converges to an homogeneous fixed point, this is also a stable density of the mean field apploximation (i.e., a rough estimate of the asymptotic density) of the corresponding Boolean CA. What is the relationship of non-homogeneous asymptotic configurations with density? The implications of the link between mean field approximation and asymptotic behaviour of FCA on Boolean CA is now under investigation.

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