Information and Computation 271 (2020) 104478

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Fault-induced dynamics of oblivious robots on a line ☆

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ARTICLE INFO

Article history:
Received 23 May 2018
Received in revised form 22 August 2019
Accepted 5 September 2019
Available online 18 September 2019

Keywords:
Distributed computing
Point convergence
Faulty agents
Oblivious robots
Look-Compute-Move model

ABSTRACT

The study of computing in presence of faulty robots in the Look-Compute-Move model has been the object of extensive investigation, typically to design fault-tolerant algorithms.
In this paper, we initiate a new line of investigation on the presence of faults, focusing on a different issue: we study, for the first time, the unintended dynamics of the robots when they execute an algorithm designed for a fault-free environment, in presence of undetectable crashed robots.
We start this investigation considering a classical point-convergence algorithm for oblivious robots with limited visibility in a simple setting (which already presents serious challenges): synchronous robots on a line with at most two faults. Interestingly, the presence of faults induces the robots to perform some form of *scattering*, rather than *point-convergence*. In fact, we discover that they arrange themselves inside the segment delimited by the two faults in complex interleaved sequences of equidistant robots.

1. Introduction

Consider a group of autonomous mobile computational entities, called *robots* that operate in a continuous space initially occupying arbitrary positions. The robots are represented as points, and they are provided with local coordinate systems (not necessarily consistent with each other) centered in themselves, and with sensor capabilities that allow them to perceive the positions of the other robots in their range of visibility. They operate in cycles of Look-Compute-Move activities: when active, a robot observes the positions of the other robots (Look), it computes a destination point (Compute), and it moves towards it (Move) [21]. Robots are *oblivious*, which means that once a cycle is completed, a robot starts the next cycle without any recollection of previous observations and computations, and the destination point is calculated solely on the basis of the current observations. Robots have no means of explicit communication (i.e., they are *silent*) and they interact only by observing each other's positions.

Robots operate synchronously (Fsync), if they perform their Look-Compute-Move activities in synchronized rounds; semi-synchronously (Ssync), if only subsets of robots are activated in synchronized rounds; or asynchronously (Async), if the activation times and the durations of each activity are totally independent.

These systems of autonomous oblivious robots have been extensively investigated under different assumptions on the various model parameters (different levels of synchrony, level of agreement on the coordinate system, visibility conditions,

☆ A preliminary version of this paper appeared in the Proceedings of SSS 2017 [14].

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[1] This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) through the Discovery Grant program.
[2] This work has been supported in part by Prof. Flocchini's University Research Chair.

https://doi.org/10.1016/j.ic.2019.104478

etc.). In these studies, the environment where they operate can be 1-dimensional (e.g., [11–13,20,26]), 2-dimensional (e.g., [9,23,25,27,29]) or, as in recent studies, 3-dimensional (e.g., [30,31]). For an overview of the recent research on the subject refer to [22] and chapters therein.

One of the most common problems studied in the Look-Compute-Move model is *pattern formation*, where the robots are initially positioned in arbitrary points and need to form a given configuration (e.g., see [23,27–29]). Particular cases of pattern formation are *gathering*, where the robots must move to occupy the same point (*point-formation*) or they have to converge towards the same point (*point-convergence*) (e.g., see [2,9,10,25]) and *circle formation*, where the robots must move to occupy equidistant points on a circle (e.g., see [17,19]). Another typical problem for mobile robots is *scattering*, where robots need to move away from each other to cover the space, typically in some regular way (e.g., see [7,8,11,18,20,24]).

Gathering, in particular, has been the object of investigation of several studies from different perspectives (computer science, control theory, artificial intelligence, biology, physics ...), and under a variety of models (continuous/discrete time, continuous/discrete sensing and movements, etc.); for a recent survey see [4].

Most algorithms in the literature are designed for fault-free groups of robots. There are several studies that consider the presence of faults, typically in the context of the gathering problem. In these studies robots might stop to operate (crash faults) or they may behave differently than intended (Byzantine faults). The goal has been to design fault-tolerant algorithms focusing on the maximum amount of faults that can be tolerated for a solution to exist in a given model (e.g., see [1,3,5,6, 8,16]). For a detailed account of the current investigations see [15].

In this paper, we consider a rather different question in presence of faulty robots that has never been asked before. Given an algorithm designed to achieve a certain global goal by a group of fault-free robots, what is the behaviour of the robots in presence of crash faults? Clearly, in most cases, the original goal is not achieved, but the theoretical interest is in characterizing the dynamics of the non-faulty robots induced by the presence of the faulty ones, from arbitrary initial configurations. Note that understanding the non-intended behaviour of a system in presence of faults might be useful also to detect the presence of faults and possibly their location; so, this study might also shed some light on fault detection in system of oblivious robots.

We start this new line of investigation focusing on the classic point-convergence algorithm by Ando et al. [2] for robots with limited visibility, and considering one of the simplest possible settings, which already proves to be challenging: fully synchronous robots (FSynch) moving in a 1-dimensional space (a line), in presence of at most two faults. In a line, the convergence algorithm prescribes each robot to move to the middle point between the two farthest visible robots on the two sides (if robots are visible on one side only, the destination is the middle between the current position and the one of the farthest visible robot). In absence of faults, starting from a configuration where the robots' "visibility graph" is connected, the robots are guaranteed to converge toward a point. It is not difficult to see that with a single fault in the system, the robots successfully converge toward the faulty robot. The presence of multiple faults, however, gives rise to intricate dynamics, and the analysis of the robots behaviour is already quite complex with just two faults.

Interestingly, and perhaps surprisingly, the presence of faults induces the robots to perform some form of *scattering*, rather than *gathering*. In fact, we discover that they converge to a hierarchical structure of interleaved sequences composed of equidistant robots, inside the segment delimited by the two faults.

Also interesting to note is the rather different dynamics that arises when moving to the middle between two robots, depending on the choice of the robots: when considering the *closest* neighbours (like in [18]), the result is an equidistant distribution, when instead selecting the *farthest* robots on the two sides the result is a much more complex structure of sequences of robots, each converging to an equidistant distribution. The main difficulty of our analysis is to show that the robots indeed form this special combination of sequences: the convergence of each sequence is then derived from a generalization of the result by [18].

The paper is organized as follows: in Section 2 we introduce the notation and the point-formation problem studied in this paper. In Section 3 we describe the scattering algorithm by Cohen and Peleg [11], and we slightly generalize the result for Fsync robots obtaining a theorem that will be needed in the rest of the paper. In Section 4 we analyze the robots' dynamics in presence of two faults: we first show that they eventually stabilize in a configuration delimited by the two faulty robots, where each robot maintains its two farthest neighbours, and where the number of points occupied by the robots does not change anymore; we then define the notion of anchored mutual chains, we prove the existence of a primary mutual chain anchored at the two faults, as well as the formation of a complex hierarchical structure of mutual chains; we finally prove convergence of all these mutual chains. We conclude in Section 5 with some open problems and research directions.

2. Preliminaries

2.1. Model and notation

Let $\mathcal{R} = \{R_0, R_1, ..., R_n\}$ denote a set of n identical robots moving on a line, simultaneously activated in synchronous time steps according to the Look-Compute-Move model. At each activation, the robots "see" the positions of the ones that are visible to them (each robot can see up to a fixed distance V), they all compute a destination point, and they move to that point. The robots are oblivious in the sense that the computation at time t solely depends on the positions of the robots perceived at that step. We assume that two robots are permanently faulty and do not participate in any activity; their faulty status, however, is not visible and they appear identical to the others. In this paper, when we need to consider an arbitrary robot in $\mathcal{R}$ but we do not need to refer to its index, we write: "let $x \in \mathcal{R}$". When we need to consider an arbitrary robot in $\mathcal{R}$ and we need to refer to its index, we write: "let $0 \le i \le n$ such that $R_i \in \mathcal{R}$".

Given a robot $x \in \mathcal{R}$, let $x(t)$ denote its position at time t with respect to the leftmost faulty robot, which is considered to be at position 0 on the real axis. Note that we use this notation for convenience, but the robots themselves do not need to have a common notion of left/right orientation. With an abuse of notation $x(t)$ may indicate both the robot itself and its position at time t. Let $\mathcal{R}(t) = \{R_0(t), R_1(t), \ldots, R_n(t)\}$ denote the *configuration* of robots at time t. Different robots do not necessarily occupy distinct positions. Indeed, there can exist a time $t \ge 0$ and two different robots $x, y \in \mathcal{R}$ such that $x(t) = y(t)$. Note however that non-faulty robots in the same position behave in the same way and can be considered as a single one. Indeed, when non-faulty robots end up in the same position, we say that they *merge* and from that moment on they will be considered as one.

Throughout the paper, we suppose that R_0 is the leftmost faulty robot and R_n is the rightmost faulty robot. Therefore, for all $t \ge 0$, we have $R_0(t) = 0$ and $R_n(t)$ is equal to some positive fixed position on the real axis.

We denote the distance between two robots $x, y \in \mathcal{R}$ at time t by $|x(t) - y(t)|$. We denote by $[\alpha, \beta]$ the interval of real numbers starting at $\alpha \in \mathbb{R}$ and ending at $\beta \in \mathbb{R}$, where $\alpha \le \beta$. Let $N(x(t))$ be the set of robots visible by x at time t, i.e.,

$$N(x(t)) = \{y(t) \in \mathcal{R}(t) \mid |x(t) - y(t)| \le V\}.$$

Let $r(x(t))$ (respectively $l(x(t))$) denote the rightmost (respectively the leftmost) visible robot from robot x at time t.

We say that a configuration of robots $\mathcal{R} = \{R_0, R_1, \ldots, R_n\}$ converges to a pattern $P = \{p_0, p_1, \ldots, p_n\}$ if for all $0 \le i \le n$, $R_i(t)$ tends to p_i as t tends to ∞, which we write $R_i(t) \overset{t\to\infty}{\longrightarrow} p_i$ throughout the paper.

2.2. Point-convergence

A classical problem for oblivious robots is gathering: the robots, initially placed in arbitrary positions, need to find themselves on the same point, not established a-priori. The convergence version of the problem requires the robots to converge toward a point. A solution to this problem is given by the well known algorithm by Ando et al. [2]. The algorithm achieves convergence to a point, not only in FSYNC systems, but also in SSYNC (i.e., when at each time step, only a subset of the robots is activated), as long as every robot is activated infinitely often.

CONVERGENCE2D [2] (for robot R_i at time t)

- $\forall R_j(t) \in N(R_i(t)) \setminus \{R_i(t)\}$,
  - $d_j(t) := dist(R_i(t), R_j(t))$,
  - $\theta_j(t) := \angle c_i(t) R_i(t) R_j(t)$,
  - $l_j(t) := (d_j(t)/2)\cos(\theta_j(t)) + \sqrt{(V/2)^2 - ((d_j(t)/2)\sin(\theta_j(t)))^2}$,
- $limit := \min_{R_j(t) \in N(R_i(t)) \setminus \{R_i(t)\}} \{l_j(t)\}$,
- $goal := dist(R_i(t), c_i(t))$,
- $D := \min\{goal, limit\}$,
- p := point on $\overline{R_i(t)c_i(t)}$ at distance D from $R_i(t)$.
- Move towards p.

Robots are initially placed in arbitrary positions in a 2-dimensional space, with limited visibility. Let $SC_i(t)$ denotes the smallest enclosing circle of the positions of robots in $\mathcal{R}(t)$ seen by $R_i(t)$; let $c_i(t)$ be the center of $SC_i(t)$. According to the algorithm, $R_i(t)$ moves toward $c_i(t)$, but only up to a certain distance. Specifically, its destination is the point p on the segment $\overline{R_i(t)c_i(t)}$ that is closest to $c_i(t)$ and that satisfies the following condition: For every robot $R_j(t) \in N(R_i(t))$, p lies in the disk $C_i(t)$ whose center is the midpoint of $R_i(t)$ and $R_j(t)$, and whose radius is $V/2$. This condition ensures that R_i and R_j will still be visible after the movement of $R_i(t)$, and possibly of $R_j(t)$.

2.2.1. The 1-dimensional case

Consider now the same algorithm in the particular case of a one-dimensional setting when the space where the robots can move is a line. In this setting, the algorithm (CONVERGENCE1D) becomes quite simple because the smallest enclosing circle of the visible robots is the segment delimited by the two farthest apart robots on the two sides, and a robot moves to occupy the mid-point between them. More precisely, if a robot sees robots only on one side, it moves to the middle between its current position and the farthest visible robot; if it sees robots on both sides, it moves to the middle between the farthest visible to the left and the farthest visible to the right.

CONVERGENCE1D (for robot R_i at time t)

- Let $l(R_i(t))$ (resp. $r(R_i(t))$) be the farthest visible robots to the left (resp. to the right). If none is visible, let $l(R_i(t)) = R_i(t)$ (resp. $r(R_i(t)) = R_i(t)$).
- Move to the midpoint between $l(R_i(t))$ and $r(R_i(t))$.

Theorem 2.1. *[2] Executing Algorithm* CONVERGENCE1D *in* FSYNCH *or* SSYNCH, *the robots converge to a point.*

3. Scattering on a segment

In [11], a classical scattering algorithm for robots in 1-dimensional systems has been analyzed both in FSYNCH and SSYNCH in a slightly different model as the one considered in this paper. A variant of this result, which is derived in Theorem 3.2 below, will be needed later.

Consider a set of oblivious robots $\mathcal{R} = \{R_0, R_1, \ldots, R_n\}$ on a line that follow the LOOK-COMPUTE-MOVE model, where R_0 and R_n do not move (equivalently, this can be considered as a segment delimited by the positions of R_0 and R_n). Let $|R_0(0), R_n(0)| = D$. The robots have *neighbouring visibility*, which means that they are able to see the two closest robots on the two sides (while R_0 and R_n know they are the delimiters of the segment). The algorithm of [11] (SPREADING) makes the robot converge to a configuration where the distance between consecutive robots tends to $\frac{D}{n}$ by having the extremal robots never move and the others move to the middle point between the two neighbouring robots.

SPREADING (for robot R_i at time step t)

- If robots are visible on one side only: do nothing.
- Let $R_i(t)^-$ and $R_i(t)^+$ be the neighbouring robots.
- Move to the midpoint between $R_i(t)^-$ and $R_i(t)^+$.

Theorem 3.1. *[11] Executing Algorithm* SPREADING *in* FSYNCH *or in* SSYNCH *on the set of robots $\mathcal{R}$ where the first and the last robots do not move, the robots converge to equidistant positions.*

We now consider the convergence result by Cohen and Peleg [11], when restricting to the FSYNCH scheduler. We prove that, in FSYNCH, convergence is achieved using the same algorithm also in a slightly more general setting. In fact, we consider the case when R_0 and R_n are not still, but they are each converging towards a point. This theorem will be needed later.

Theorem 3.2. *Let $\mathcal{R} = \{R_0, R_1, \ldots, R_n\}$, where $R_0(t) \stackrel{t\to\infty}{\longrightarrow} R'_0$ and $R_n(t) \stackrel{t\to\infty}{\longrightarrow} R'_n$. Executing Algorithm* SPREADING *in* FSYNCH *on the set of robots $\{R_1, \ldots, R_{n-1}\}$, the robots converge to equidistant positions between R'_0 and R'_n.*

Proof. Without loss of generality, suppose that $R'_0 = 0$ and $R'_n = 1$. We want to prove that $R_i(t) \stackrel{t\to\infty}{\longrightarrow} \frac{i}{n}$ for all $1 \le i \le n-1$. We follow the proof of Theorem 3.1. For all $1 \le i \le n-1$, the next position of $R_i(t)$ is

$$R_i(t+1) = \frac{R_{i-1}(t) + R_{i+1}(t)}{2}.$$

Let $\eta_i(t) = R_i(t) - \frac{i}{n}$ for all $0 \le i \le n$. We get

$$\eta_i(t+1) = \frac{\eta_{i-1}(t) + \eta_{i+1}(t)}{2}$$

for all $1 \le i \le n-1$. Our goal is to show that $\eta_i(t) \stackrel{t\to\infty}{\longrightarrow} 0$ for all $0 \le i \le n$. By the hypothesis, we already know that $\eta_0(t) \stackrel{t\to\infty}{\longrightarrow} 0$ and $\eta_n(t) \stackrel{t\to\infty}{\longrightarrow} 0$. The fact that $\eta_i(t) \stackrel{t\to\infty}{\longrightarrow} 0$ for all $1 \le i \le n-1$ relies on the following lemma. □

Lemma 3.3. *Let $m \ge 2$ be an integer. Consider a sequence $\eta_i(t)$ of real numbers, where $0 \le i \le m$ and $t \ge 0$. Suppose that $\eta_0(t) \stackrel{t\to\infty}{\longrightarrow} 0$ and $\eta_m(t) \stackrel{t\to\infty}{\longrightarrow} 0$, and that*

$$\eta_i(t+1) = \frac{\eta_{i-1}(t) + \eta_{i+1}(t)}{2}$$

for all $1 \le i \le m-1$ and $t \ge 0$. Moreover, suppose that there exists a positive real number M such that $|\eta_i(t)| \le M$ for all $0 \le i \le m$ and $t \ge 0$. Then, for all $0 \le i \le m$, $\eta_i(t) \stackrel{t\to\infty}{\longrightarrow} 0$.

The details of some calculations in this proof are technical. We present a proof sketch below and postpone the full proof to Appendix A.

Proof. By the hypothesis, the lemma is true for $i = 0$ and $i = m$. If $m = 2$, by the hypothesis,

$$|\eta_1(t+1)| = \left|\frac{\eta_0(t) + \eta_2(t)}{2}\right| \leq \frac{|\eta_0(t)| + |\eta_2(t)|}{2} \overset{t\to\infty}{\longrightarrow} 0.$$

Hence, assume that $m \geq 3$. To deal with other values of i, let

$$\psi(t) = \sum_{i=0}^{m} \eta_i^2(t).$$

We show that $\psi(t) \overset{t\to\infty}{\longrightarrow} 0$, which completes the proof. Following the same approach as the one used in the proof of Theorem 3.1, we use the Fourier sine series of $\eta_i(t)$. However, in our case, we need to be careful since $\eta_0(t)$ and $\eta_m(t)$ are not necessarily equal to 0. For all $0 \leq i \leq m$, $0 \leq k \leq m$ and $t \geq 0$, let

$$g(i,k) = \sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \qquad \text{and} \qquad \mu_k(t) = \sum_{i=0}^{m} \eta_i(t) g(i,k).$$

We have

$$\eta_i(t) = \sum_{k=0}^{m} \mu_k(t) g(i,k) \tag{1}$$

for all $1 \leq i \leq m-1$ and $t \geq 0$. Moreover, we have

$$g(i,0) = g(i,m) = g(0,k) = g(m,k) = 0 \tag{2}$$

for all $0 \leq i \leq m$ and $0 \leq k \leq m$, from which

$$\sum_{k=0}^{m} \mu_k(t) g(0,k) = \sum_{k=0}^{m} \mu_k(t) g(m,k) = 0.$$

Observe that for all $0 \leq k \leq m$ and $0 \leq q \leq m$,

$$\sum_{i=0}^{m} g(i,k) g(i,q) = \begin{cases} 0 & \text{if } k = 0, k = m, q = 0 \text{ or } q = m \\ \delta_{k,q} & \text{otherwise,} \end{cases} \tag{3}$$

where $\delta_{k,q}$ stands for the Kronecker's delta, i.e., $\delta_{k,q} = 1$ if $k = q$ and 0 otherwise. Moreover, observe that for all $0 \leq k \leq m$ and $t \geq 0$,

$$|\mu_k(t)| = \left|\sum_{i=0}^{m} \eta_i(t) g(i,k)\right| = \left|\sum_{i=1}^{m-1} \eta_i(t) g(i,k)\right| \leq \sum_{i=1}^{m-1} |\eta_i(t) g(i,k)| \leq (m-1)\sqrt{\frac{2}{m}} M \leq \sqrt{2m} M.$$

We can show that

$$\psi(t) = \eta_0^2(t) + \eta_m^2(t) + \sum_{k=1}^{m-1} \mu_k^2(t).$$

Moreover, for $1 \leq k \leq m-1$, we can show that

$$\mu_k(t+1) \leq |\eta_0(t)| + |\eta_m(t)| + \cos\left(\frac{k\pi}{m}\right) \mu_k(t).$$

Therefore, we can show that

$$\begin{aligned} \psi(t+1) \leq\ & \eta_0^2(t+1) + \eta_m^2(t+1) + (m-1)\left(|\eta_0(t)| + |\eta_m(t)|\right)^2 \\ & + 2\left(|\eta_0(t)| + |\eta_m(t)|\right)(m-1)\sqrt{2m} M + \cos^2\left(\frac{\pi}{m}\right) \psi(t). \end{aligned} \tag{4}$$

Since $\eta_0(t) \stackrel{t\to\infty}{\longrightarrow} 0$ and $\eta_m(t) \stackrel{t\to\infty}{\longrightarrow} 0$, there exists a positive function $\epsilon(t)$ such that $\epsilon(t) \stackrel{t\to\infty}{\longrightarrow} 0$, $\epsilon(t) = O(1)$ and for all $t \geq 0$, we have $|\eta_0(t)| \leq \epsilon(t)$, $|\eta_0(t+1)| \leq \epsilon(t)$, $|\eta_m(t)| \leq \epsilon(t)$ and $|\eta_m(t+1)| \leq \epsilon(t)$. Therefore, the right-hand side of inequality (4) is upper bounded by

$$\begin{aligned}
&\epsilon^2(t) + \epsilon^2(t) + 4(m-1)\epsilon^2(t) + 4\epsilon(t)(m-1)\sqrt{2m}M + \cos^2\left(\frac{\pi}{m}\right)\psi(t) \\
&\leq \left(2\epsilon(t) + 4(m-1)\epsilon(t) + 4(m-1)\sqrt{2m}M\right)\epsilon(t) + \cos^2\left(\frac{\pi}{m}\right)\psi(t) \\
&\leq \phi\,\epsilon(t) + \lambda\,\psi(t),
\end{aligned}$$

where $\phi = 2\epsilon(t) + 4(m-1)\epsilon(t) + 4(m-1)\sqrt{2m}M = O(1)$ and $\lambda = \cos^2\left(\frac{\pi}{m}\right) \leq \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$.

Consequently,

$$\begin{aligned}
\psi(t) &\leq \left(1 + \lambda + \lambda^2 + \ldots + \lambda^{t-1}\right)\phi\,\epsilon(t) + \lambda^t\psi(0) \\
&= \frac{1-\lambda^t}{1-\lambda}\phi\,\epsilon(t) + \lambda^t\psi(0) \\
&\stackrel{t\to\infty}{\longrightarrow} 0. \quad \square
\end{aligned}$$

4. Robots' dynamics

We now consider Algorithm CONVERGENCE1D, when the robots are initially placed in arbitrary points of a line. We remind that, in Algorithm CONVERGENCE1D, at time t robot $x(t)$ observes the position of all robots at distance at most V and it moves to occupy the middle point between the right-most visible robots $r(x(t))$ and the left-most visible robot $l(x(t))$. If robots are visible on one side only, $x(t)$ moves to the middle point between itself and the farthest visible robot. Note that the indication of left and right is used for convenience, but a consistent left/right orientation is not needed for the robots, which simply consider for their computation the farthest robots on the two sides.

It is easy to see that if the configuration contains a single faulty robot, the other robots converge toward it. In this Section we then focus on the case when the system contains two faults and we show that, starting from an arbitrary configuration and following algorithm CONVERGENCE1D, the system converges towards a limit configuration.

4.1. Basic properties

We start with a series of lemmas leading to the proof of two crucial properties: there exists a time after which robots preserve their farthest neighbours (Theorem 4.8) and there exists a time after which the number of different positions occupied by them becomes constant (Corollary 4.7).

Lemma 4.1 *(No crossing). If $x, y \in \mathcal{R}$ are two non-faulty robots and $x(t) < y(t)$, then $x(t+1) \leq y(t+1)$.*

Proof. Since $x(t) < y(t)$, we have that $r(x(t)) \leq r(y(t))$ and $l(x(t)) \leq l(y(t))$ by definition. It follows that $x(t+1) = \frac{l(x(t))+r(x(t))}{2} \leq \frac{l(y(t))+r(y(t))}{2} = y(t+1)$. $\square$

With the next three lemmas (4.2, 4.3, and 4.4), we show that all robots, except possibly two, eventually enter the segment $[R_0, R_n]$ delimited by the two faulty robots. At most two robots might perpetually stay outside of it, one to the left of R_0 and one to the right of R_n. If this is the case, however, the two outsiders converge to R_0 and R_n, respectively.

Lemma 4.2. *Either one of the following two scenarios happens as $t \to \infty$.*

1. *In a finite number of steps, all robots place themselves inside the line segment $[R_0, R_n]$ and stay inside the line segment $[R_0, R_n]$.*
2. *There is at least one robot x that never enters the line segment $[R_0, R_n]$. If $x(0) < R_0$, then $x(t) \stackrel{t\to\infty}{\longrightarrow} R_0$. If $x(0) > R_n$, then $x(t) \stackrel{t\to\infty}{\longrightarrow} R_n$.*

Proof. Since the two faulty robots do not move, they are already inside $[R_0, R_n]$. For the rest of the proof, we consider only the non-faulty robots. Let x_ℓ and x_r be the leftmost and the rightmost non-faulty robots, respectively.

1. By Lemma 4.1, x_ℓ (respectively x_r) stays the leftmost (respectively the rightmost) non-faulty robot at all steps of the execution of the algorithm. Therefore, it is sufficient to prove the lemma for x_ℓ and x_r.
   We first argue that if at some time $t_0 > 0$, $x_\ell(t_0) \in [R_0, R_n]$, then for all $t > t_0$, $x_\ell(t) \geq R_0$. Since x_ℓ is the leftmost non-faulty robot and $x_\ell(t_0) \in [R_0, R_n]$, we have $l(x_\ell(t_0)) \geq R_0$. Therefore,

$$x_\ell(t_0+1) = \frac{l(x_\ell(t_0)) + r(x_\ell(t_0))}{2} \geq \frac{R_0 + x_\ell(t_0)}{2} \geq \frac{R_0+R_0}{2} = R_0,$$

from which the proof follows by induction on t. A symmetric argument shows that if $x_r(t_0) \in [R_0, R_n]$, then for all $t > t_0$, $x_r(t) \leq R_n$. It remains to consider the case where x_ℓ or x_r never enters $[R_0, R_n]$.

2. Suppose that x_ℓ does not enter the interval $[R_0, R_n]$ in a finite number of steps. Therefore,[3] $x_\ell(t) < R_0$ for all $t \geq 0$. Together with the fact that x_ℓ is the leftmost non-faulty robot, we get $l(x_\ell(t)) = x_\ell(t)$ and $r(x_\ell(t)) > x_\ell(t)$ for all $t \geq 0$. Therefore,

$$x_\ell(t+1) = \frac{l(x_\ell(t)) + r(x_\ell(t))}{2} > \frac{x_\ell(t) + x_\ell(t)}{2} = x_\ell(t)$$

for all $t \geq 0$. It follows that $x_\ell(t)$ is strictly increasing for $t \geq 0$. Since $x_\ell(t) < R_0$ for all $t \geq 0$, $x_\ell(t)$ converges to a point $x_\ell^* \leq R_0$ as $t \to \infty$. Observe that $x_\ell(t) < x_\ell^*$ for all $t \geq 0$.

Since $x_\ell(t) \overset{t\to\infty}{\longrightarrow} x_\ell^*$, there is a time $t' > 0$ such that $x_\ell^* - x_\ell(t) < \frac{V}{4}$ for all $t \geq t'$. We claim that $r(x_\ell(t)) = R_0$ for all $t \geq t'$. Assume this is true. Since $l(x_\ell(t)) = x_\ell(t)$ for all $t \geq t'$, then

$$x_\ell(t+1) = \frac{l(x_\ell(t)) + r(x_\ell(t))}{2} = \frac{x_\ell(t) + R_0}{2} = \frac{x_\ell(t) + 0}{2} = \frac{x_\ell(t)}{2}$$

for all $t \geq t'$. This shows that $x_\ell(t) \overset{t\to\infty}{\longrightarrow} R_0 = 0$, i.e., $x_\ell^* = R_0$.

We prove our claim by contradiction. Suppose that there is a time $t_0 \geq t'$ such that $r(x_\ell(t_0)) \neq R_0$. Let $\delta = x_\ell^* - x_\ell(t_0) < \frac{V}{4}$. Let $x_1(t_0) = r(x_\ell(t_0)) \neq R_0$ and $\delta' = x_1(t_0) - x_\ell^*$. We do not know whether $x_1(t_0)$ is to the left or to the right of x_ℓ^*, i.e., we do not know the sign of δ'. Since $x_\ell(t)$ is strictly increasing and $x_\ell(t) < x_\ell^*$ for all $t \geq 0$, we have $|\delta'| < \delta$. Moreover, $x_1(t_0) - x_\ell(t_0) = \delta + \delta'$ and $x_\ell^* - x_\ell(t_0+1) = \frac{\delta - \delta'}{2}$. We now look at the rightmost visible robot from $x_1(t_0)$. We have $r(x_1(t_0)) - x_\ell(t_0) > V$, otherwise $x_1(t_0)$ would not be the rightmost visible robot from $x_\ell(t_0)$. Therefore, we have

$$r(x_1(t_0)) - x_1(t_0) = (r(x_1(t_0)) - x_\ell(t_0)) + (x_\ell(t_0) - x_1(t_0)) > V - (\delta + \delta'). \tag{5}$$

We also have

$$\begin{aligned} x_1(t_0+1) - x_\ell(t_0+1) &= \frac{l(x_1(t_0)) + r(x_1(t_0))}{2} - \frac{l(x_\ell(t_0)) + r(x_\ell(t_0))}{2} \\ &= \frac{x_\ell(t_0) + r(x_1(t_0))}{2} - \frac{x_\ell(t_0) + x_1(t_0)}{2} \\ &= \frac{r(x_1(t_0)) - x_1(t_0)}{2} \\ &< V, \end{aligned} \tag{6}$$

from which $x_1(t_0+1)$ is visible from $x_\ell(t_0+1)$. This leads to

$$\begin{aligned} x_\ell(t_0+2) - x_\ell^* &= \frac{l(x_\ell(t_0+1)) + r(x_\ell(t_0+1))}{2} - x_\ell^* \\ &\geq \frac{x_\ell(t_0+1) + x_1(t_0+1)}{2} - x_\ell^* \\ &= \frac{(x_\ell(t_0+1) - x_\ell^*) + (x_1(t_0+1) - x_\ell^*)}{2} \\ &= \frac{(x_\ell(t_0+1) - x_\ell^*) + (x_1(t_0+1) - x_\ell(t_0+1)) + (x_\ell(t_0+1) - x_\ell^*)}{2} \\ &= \frac{2(x_\ell(t_0+1) - x_\ell^*) + (x_1(t_0+1) - (x_\ell(t_0+1))}{2} \\ &> \frac{(\delta' - \delta) + \frac{V-(\delta+\delta')}{2}}{2} && \text{from (6) and (5)} \\ &= \frac{V - 3\delta + \delta'}{4} \\ &> \frac{V - 4\delta}{4} \\ &> 0, \end{aligned}$$

[3] The case where x_ℓ is to the right of R_n is taken care of by the case where x_r is to the right of R_n.

from which $x_\ell(t_0+2) > x_\ell^*$, which is a contradiction since $x_\ell(t) < x_\ell^*$ for all $t \geq 0$. This completes the proof of our claim. A symmetric argument holds for x_r. □

Lemma 4.3 *(No more crossing). If x is a non-faulty robot, it will cross at most a finite number of times with a faulty robot.*

Proof. Let x_ℓ be the leftmost non-faulty robot. From Lemma 4.1, x_ℓ will stay the leftmost non-faulty robot at all steps of the execution of the algorithm. Moreover, from Lemma 4.2, two scenarios are possible: after some time t_0, (1) x_ℓ enters the line segment $[R_0, R_n]$ and for all $t \geq t_0$, $x_\ell(t) \geq R_0$ or (2) $x_\ell(t)$ is strictly increasing for $t \geq 0$ and $x_\ell(t) \overset{t\to\infty}{\longrightarrow} R_0 = 0$.

1. In this case, after time t_0, no robot will cross R_0.
2. In this case, let x be a robot and $t' \geq 0$ be a time such that $x(t') > 0$, $x(t'+1) < 0$ and $R_0 - x_\ell(t') = 0 - x_\ell(t') < \frac{V}{2}$. Since x_ℓ is the leftmost non-faulty agent, we have
$$l(x(t')) \geq x_\ell(t'). \tag{7}$$
Moreover, since $x(t'+1) < R_0 = 0$, we have
$$x(t'+1) = \frac{l(x(t')) + r(x(t'))}{2} < R_0 = 0. \tag{8}$$
Therefore, from (8) and (7), we get
$$r(x(t')) - x_\ell(t') < (R_0 - l(x(t'))) + (R_0 - x_\ell(t')) \leq 2(R_0 - x_\ell(t')) < V. \tag{9}$$
Thus, $r(x_\ell(t')) = r(x(t'))$ otherwise $r(x(t'))$ would not be the rightmost visible robot from $x(t')$.
Also, since x_ℓ is the leftmost non-faulty agent, we have
$$0 < x(t') - x_\ell(t') \leq r(x(t')) - x_\ell(t') < V$$
by (9). Thus, $l(x(t')) = x_\ell(t') = l(x_\ell(t'))$.
Since $r(x(t')) = r(x_\ell(t'))$ and $l(x(t')) = l(x_\ell(t'))$, we have that $x(t'+1) = x_\ell(t'+1)$, i.e., x and x_ℓ merge at step $t'+1$. Since $x_\ell(t)$ is strictly increasing for $t \geq 0$ and $x_\ell(t) \overset{t\to\infty}{\longrightarrow} R_0 = 0$, x will not cross $R_0 = 0$ anymore.

A symmetric argument with the rightmost non-faulty robot x_r completes the proof. □

Lemma 4.4 *(At most two outsiders). Let x_ℓ (respectively x_r) be the leftmost (respectively the rightmost) non-faulty robot.*

- *If x_ℓ never enters the line segment $[R_0, R_n]$, then after a finite number of steps, x_ℓ is the only robot to the left of R_0.*
- *If x_r never enters the line segment $[R_0, R_n]$, then after a finite number of steps, x_r is the only robot to the right of R_n.*

Proof. Assume that there is a robot x and a time $t \geq 0$ such that $x_\ell(t) < x(t) < R_0 = 0$ (the case where $R_n < x(t) < x_r(t)$ is symmetric). Moreover, assume that $R_0 - x_\ell(t) < \frac{V}{2}$ and $x_r(t) - R_n < \frac{V}{2}$. (Observe that if $x_r(t) \leq R_n$, then $x_r(t) - R_n \leq 0 < \frac{V}{2}$.) We consider two cases: (1) x eventually enters the line segment $[R_0, R_n]$ or (2) not.

1. If x enters the line segment $[R_0, R_n]$ and stays there, then after a finite number of steps, it is not outside of the line segment $[R_0, R_n]$. Therefore, let us consider the case where x enters the line segment $[R_0, R_n]$ and eventually gets out of $[R_0, R_n]$. If x gets out of $[R_0, R_n]$ by crossing R_0, then it merges with x_ℓ (refer to the proof of Lemma 4.3). If x gets out of $[R_0, R_n]$ by crossing R_n, then it merges with x_r (refer to the proof of Lemma 4.3).
2. Assume that x never enters the line segment $[R_0, R_n]$. This part of the proof is similar to the proof of Lemma 4.3. Since x_ℓ is the leftmost non-faulty agent, we have
$$l(x(t)) \geq x_\ell(t). \tag{10}$$
Moreover, since x never enters the line segment $[R_0, R_n]$, we have
$$x(t+1) = \frac{l(x(t)) + r(x(t))}{2} < R_0 = 0. \tag{11}$$
Therefore, from (11) and (10), we get
$$r(x(t)) - x_\ell(t) < (R_0 - l(x(t))) + (R_0 - x_\ell(t)) \leq 2(R_0 - x_\ell(t)) < V. \tag{12}$$
Thus, $r(x_\ell(t)) = r(x(t))$ otherwise $r(x(t))$ would not be the rightmost visible robot from $x(t)$.

Also, since x_ℓ is the leftmost non-faulty agent, we have

$$0 < x(t) - x_\ell(t) \le r(x(t)) - x_\ell(t) < V$$

by (12). Thus, $l(x(t)) = x_\ell(t) = l(x_\ell(t))$.
Since $r(x(t)) = r(x_\ell(t))$ and $l(x(t)) = l(x_\ell(t))$, we have that $x(t+1) = x_\ell(t+1)$, i.e., x and x_ℓ merge at step $t+1$.

In all cases, if there is a robot x between x_ℓ and R_0, then after a finite number of steps, x enters the line segment $[R_0, R_n]$ and stays there or x merges with another robot. Since we have a finite number of robots, after a finite number of steps, x_ℓ will be the only robot satisfying $x_\ell < R_0$. □

The following corollary states that if x_ℓ (respectively x_r) never enters $[R_0, R_n]$, then after a finite number of steps, it will not interact with any other robots. Hence, we can ignore it.

Corollary 4.5. *Let x_ℓ (respectively x_r) be the leftmost (respectively the rightmost) non-faulty robot.*

- *If x_ℓ never enters $[R_0, R_n]$, then after a finite number of steps, x_ℓ only sees itself and R_0. Moreover, only x_ℓ and R_0 see x_ℓ.*
- *If x_r never enters $[R_0, R_n]$, then after a finite number of steps, x_r only sees itself and R_n. Moreover, only x_r and R_n see x_r.*

Proof. From Lemma 4.4, after a finite number of steps, x_ℓ will be the only non-faulty agent to the left of R_0. Let $t \ge 0$ be any time such that $R_0 - x_\ell(t) < \frac{V}{4}$. From the proof of Lemma 4.2, we know that $r(x_\ell(t)) = R_0$. Hence, x_ℓ only sees itself and R_0, from which we know that x_ℓ and R_0 see x_ℓ. Assume that a non-faulty agent $x \in [R_0, R_n]$ sees x_ℓ at time t. Then $x(t) - x_\ell(t) \le V$, from which $x_\ell(t)$ sees $x(t)$. This contradicts the fact that $r(x_\ell(t)) = R_0$.
A symmetric argument holds for x_r. □

We now show that during the evolution of the system, a robot never loses visibility of the robots seen in the past. This proves that even in the presence of faulty robots, starting from a configuration where the robots' visibility graph is connected, the robots' visibility graph stays connected.

Lemma 4.6 *(Preserved visibility). Let $t \ge 0$ be an arbitrary time and $y \in N(x(t))$. For all $t' > t$, $y \in N(x(t'))$.*

Proof. Let $y \in N(x(t))$. Hence, we have $|x(t) - y(t)| \le V$. Without loss of generality, suppose that $y(t)$ is to the left of $x(t)$, from which $0 < x(t) - y(t) \le V$. We consider three cases: (1) x and y are non-faulty, (2) exactly one of x and y is faulty, or (3) both x and y are faulty.

1. In this case, by Lemma 4.1, $x(t+1) - y(t+1) \ge 0$. We have

$$\begin{aligned} x(t+1) - y(t+1) &= \frac{l(x(t)) + r(x(t))}{2} - \frac{l(y(t)) + r(y(t))}{2} \\ &\le \frac{y(t) + (x(t) + V)}{2} - \frac{(y(t) - V) + x(t)}{2} \\ &= V. \end{aligned}$$

2. Without loss of generality, suppose that x is faulty and y is non-faulty. If $x(t+1) - y(t+1) \ge 0$, we have

$$\begin{aligned} x(t+1) - y(t+1) &= x(t) - \frac{l(y(t)) + r(y(t))}{2} \\ &\le x(t) - \frac{(y(t) - V) + x(t)}{2} \\ &= \frac{x(t) - y(t) + V}{2} \\ &\le \frac{V + V}{2} \\ &= V. \end{aligned}$$

If $y(t+1) - x(t+1) \ge 0$, we have

$$\begin{aligned} y(t+1) - x(t+1) &= \frac{l(y(t)) + r(y(t))}{2} - x(t) \\ &\le \frac{x(t) + (y(t) + V)}{2} - x(t) \end{aligned}$$

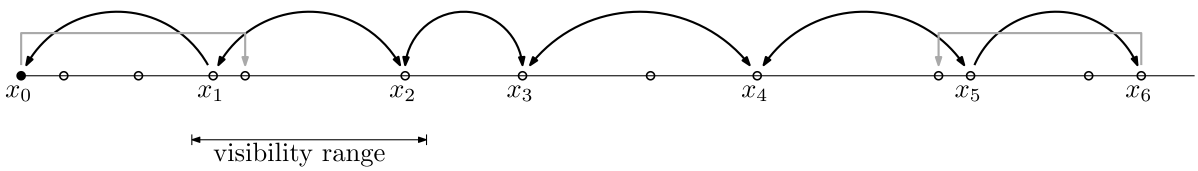

Fig. 1. A mutual chain of robots $C(t) = \{x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)\}$ anchored in x_0 and x_6, where the arrows indicate farthest visibility.

$$
\begin{aligned}
&= \frac{y(t) - x(t) + V}{2} \\
&\le \frac{0 + V}{2} \\
&< V .
\end{aligned}
$$

3. In this case, we have $x(t+1) = x(t)$ and $y(t+1) = y(t)$, so the result follows. □

A robot never loses visibility of the robots seen in the past; however, notice that new robots could enter its visibility range (*inclusion*). It is also possible for robots to merge and occupy the same position (*merging*). Once some robots occupy the same position they act as one single robot (except possibly for a non-faulty robot merging with a faulty one).

Definition 1 *(Size-stable time).* A time t_0 is called a *size-stable time* if, for all $t \ge t_0$,

- there will be no inclusions, mergings or crossings in the system,
- and either all agents are inside the line segment $[R_0, R_n]$ or at most one agent is on each side of the line segment $[R_0, R_n]$ and stay outside of $[R_0, R_n]$. Moreover, the two outsiders converge to R_0 and R_n, respectively.

Observe that if t_0 is a size-stable time, then t is a size-stable time for all $t \ge t_0$.

From Lemmas 4.1 and 4.3, after a finite number of steps, no two robots are *crossing* each others. From Lemma 4.4, either all robots are inside the line segment $[R_0, R_n]$ after a finite number of steps, or at most two robots will stay outside of the line segment $[R_0, R_n]$ for all time $t \ge 0$. We then get the following corollary.

Corollary 4.7. *For all sets of robots, there exists a size-stable time t_0.*

Finally, from Lemmas 4.1, 4.3 and 4.6, and Corollary 4.7, we can conclude that at any time after a size-stable time t is reached, the farthest left and right neighbours, namely $l(x(t))$ and $r(x(t))$, of any robot x will never change.

Theorem 4.8 *(Preserved-farthest-neighbours). Let t be a size-stable time and $x \in \mathcal{R}$ be a robot. For all $t' > t$, $r(x(t')) = r(x(t))$ and $l(x(t')) = l(x(t))$.*

For the rest of the paper, we suppose that the earliest size-stable time is 0. Thus, from Corollary 4.7, for all $t \ge 0$, t is a size-stable time.

4.2. Convergence of mutual chains

We now define the notion of *mutual chain* as a set of robots that are mutually the farthest from each other.

Definition 2 *(Mutual chain).* Let $0 \le k \le n$ be an integer and $t \ge 0$ be any size-stable time. A *mutual chain at time t* (or *mutual chain* for short) is a configuration $C(t) = \{x_0(t), x_1(t), ..., x_k(t)\} \subseteq \mathcal{R}(t)$ made of $k+1$ robots such that for all $0 \le i \le k-1$, $l(x_{i+1}(t)) = x_i(t)$ and $r(x_i(t)) = x_{i+1}(t)$ (refer to Fig. 1).

If $r(x_i(t)) = x_j(t)$ and $l(x_j(t)) = x_i(t)$, we say that x_i and x_j are *mutually chained at time t* or that $x_i(t)$ and $x_j(t)$ are *mutually chained.*

Observe that in a mutual chain, the difference between two consecutive robots is more than $\frac{V}{2}$. The *anchors* of a mutual chain $C(t) = \{x_0(t), x_1(t), ..., x_k(t)\}$ are the farthest left neighbour of $x_0(t)$ and the farthest right neighbour of $x_k(t)$.

Definition 3 *(Anchors).* Given a mutual chain $C(t) = \{x_0(t), x_1(t), \ldots, x_k(t)\}$, we say that $l(x_0(t))$ and $r(x_k(t))$ are the left and right *anchors* of $C(t)$ (or that $C(t)$ is *anchored* at $l(x_0(t))$ and $r(x_k(t))$) (refer to Fig. 1).

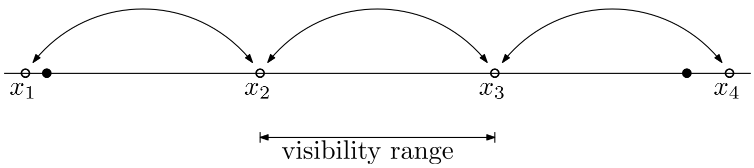

Fig. 2. The configuration $\{x_1, x_2, x_3, x_4\}$ is a mutual chain. It is anchored at x_1 and x_4.

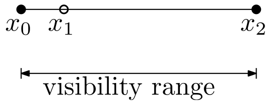

Fig. 3. The configuration $\{x_1\}$ is a mutual chain. It is anchored at x_0 and x_2.

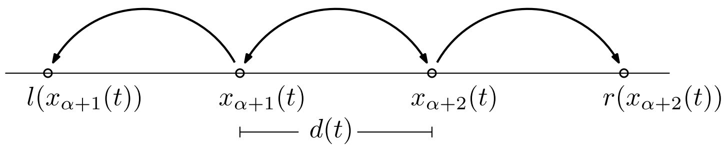

Fig. 4. Illustration of Lemma 4.10.

Note that the definition of anchor allows the anchors of a mutual chain to be part of the mutual chain (refer to Fig. 2). Moreover, the definition of mutual chain allows a mutual chain to possibly contain only one robot (refer to Fig. 3). Note that the anchors do not have to be faulty robots for this situation to happen. Indeed, any robot x forms a mutual chain $\{x(t)\}$ anchored at $l(x(t))$ and $r(x(t))$.

We now prove the formation, during the execution of the algorithm, of a special unique mutual chain called *primary chain*. The primary chain is a mutual chain starting from R_0 and ending in R_n. We will then introduce a hierarchical notion of mutual chains with different levels, where chains of some level are anchored in lower level ones. Intuitively, if a robot remains strictly between two consecutive robots of a given chain, then it belongs to a chain which is anchored in a lower level chain. Moreover, we will show that the robots will eventually arrange themselves in such a hierarchical structure of mutual chains.

Let us first prove the existence of the primary chain.

Theorem 4.9 *(Primary chain). There exists a configuration of robots $\mathcal{C}_1 = \{x_0, x_1, \ldots, x_k\} \subseteq \mathcal{R}$ such that at any size-stable time $t > 0$, $\mathcal{C}_1(t)$ is a mutual chain anchored at R_0 and R_n, where $x_0 = R_0$ and $x_k = R_n$. This mutual chain is called the* primary chain *of $\mathcal{R}$ and it is unique.*

Before we prove Theorem 4.9, we need the following technical lemma. Intuitively, the lemma states that when the distance between two mutually chained robots tends to V (as $t \to \infty$), this limit behaviour propagates to the leftmost and rightmost visible robots.

Lemma 4.10. *Let $x_{\alpha+1}, x_{\alpha+2} \in \mathcal{R}$ such that for all $t \geq 0$ (refer to Fig. 4),*

- *$x_{\alpha+1}(t)$ and $x_{\alpha+2}(t)$ are mutually chained,*
- *$d(t) = (x_{\alpha+2}(t) - x_{\alpha+1}(t)) \stackrel{t\to\infty}{\longrightarrow} V$,*
- *$l(x_{\alpha+1}(t)) \neq x_{\alpha+1}(t)$*
- *and $r(x_{\alpha+2}(t)) \neq x_{\alpha+2}(t)$.*

Then $r(x_{\alpha+2}(t)) - x_{\alpha+2}(t) \stackrel{t\to\infty}{\longrightarrow} V$ and $x_{\alpha+1}(t) - l(x_{\alpha+1}(t)) \stackrel{t\to\infty}{\longrightarrow} V$.

Proof. We have

$$x_{\alpha+1}(t+1) = \frac{l(x_{\alpha+1}(t)) + r(x_{\alpha+1}(t))}{2} = \frac{l(x_{\alpha+1}(t)) + x_{\alpha+1}(t) + d(t)}{2} \tag{13}$$

and

$$\begin{aligned} x_{\alpha+2}(t+1) &= \frac{l(x_{\alpha+2}(t)) + r(x_{\alpha+2}(t))}{2} \\ &= \frac{x_{\alpha+1}(t) + r(x_{\alpha+2}(t))}{2} \\ &= \frac{x_{\alpha+1}(t) + r(x_{\alpha+2}(t)) + d(t) - d(t)}{2} \\ &= \frac{x_{\alpha+1}(t) + r(x_{\alpha+2}(t)) + d(t) - (x_{\alpha+2}(t) - x_{\alpha+1}(t))}{2} \\ &= \frac{2x_{\alpha+1}(t) - x_{\alpha+2}(t) + r(x_{\alpha+2}(t)) + d(t)}{2}. \end{aligned} \tag{14}$$

Since $x_{\alpha+1}$ and $x_{\alpha+2}$ are mutually chained and $d(t) \overset{t\to\infty}{\longrightarrow} V$, there is a function $\epsilon(t)$ such that $\epsilon(t) \overset{t\to\infty}{\longrightarrow} 0$ and

$$d(t+1) = x_{\alpha+2}(t+1) - x_{\alpha+1}(t+1) > V - \epsilon(t).$$

Consequently,

$$\begin{aligned} & d(t+1) \\ &= x_{\alpha+2}(t+1) - x_{\alpha+1}(t+1) \\ &= \left(\frac{2x_{\alpha+1}(t) - x_{\alpha+2}(t) + r(x_{\alpha+2}(t)) + d(t)}{2}\right) - \left(\frac{l(x_{\alpha+1}(t)) + x_{\alpha+1}(t) + d(t)}{2}\right) \qquad \text{by (13) and (14)} \\ &= \frac{x_{\alpha+1}(t) - l(x_{\alpha+1}(t)) + r(x_{\alpha+2}(t)) - x_{\alpha+2}(t)}{2} \\ &> V - \epsilon(t). \end{aligned} \tag{15}$$

Let $\delta_1(t)$ and $\delta_2(t)$ be two functions such that $V - \delta_1(t) = x_{\alpha+1}(t) - l(x_{\alpha+1}(t))$ and $V - \delta_2(t) = r(x_{\alpha+2}(t)) - x_{\alpha+2}(t)$. Since $l(x_{\alpha+1}(t)) \neq x_{\alpha+1}(t)$ and $r(x_{\alpha+2}(t)) \neq x_{\alpha+2}(t)$ for all $t \geq 0$, we have $0 \leq \delta_1(t) < V$ and $0 \leq \delta_2(t) < V$. Therefore, from (15), we get

$$\frac{V - \delta_1(t) + V - \delta_2(t)}{2} > V - \epsilon(t),$$

from which

$$0 \leq \frac{\delta_1(t) + \delta_2(t)}{2} < \epsilon(t) \overset{t\to\infty}{\longrightarrow} 0.$$

This means that $\delta_1(t) \overset{t\to\infty}{\longrightarrow} 0$ and $\delta_2(t) \overset{t\to\infty}{\longrightarrow} 0$, from which $x_{\alpha+1}(t) - l(x_{\alpha+1}(t)) \overset{t\to\infty}{\longrightarrow} V$ and $r(x_{\alpha+2}(t)) - x_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$. □

Proof of Theorem 4.9. [Uniqueness] We first explain that if the primary chain exists, then it is unique. Since $R_0 = x_0$ and $R_n = x_k$ are part of the mutual chain, if we start at R_0, we get $x_1 = r(R_0)$ and $x_{i+1} = r(x_i)$ for all $0 \leq i \leq k-1$, where $x_k = R_n$. So each x_i is uniquely defined.

[Existence] We now prove that the primary chain does exist. By Lemma 4.4, at any size-stable time t, there is at most one robot x_ℓ to the left of R_0 which will never enter $[R_0, R_n]$ and there is at most one robot x_r to the right of R_n which will never enter $[R_0, R_n]$. Moreover, $x_\ell \overset{t\to\infty}{\longrightarrow} R_0$ and $x_r \overset{t\to\infty}{\longrightarrow} R_n$. From Corollary 4.5, after a finite number of steps, x_ℓ and x_r will not interact with any other robots. Hence, without loss of generality, we can assume that $\mathcal{R}(t) \subset [R_0, R_n]$ for any size-stable time t. We need to prove that the primary chain exists.

We prove the existence of the primary chain by contradiction. Therefore, assume that there does not exist any mutual chain. Let us start by summarizing the steps of the proof. 1) We construct a particular configuration, composed by a forward-chain from R_0 and connecting each node to its farthest right neighbour until R_n is reached and a backward chain from R_n connecting each node to its farthest left neighbour back to R_0. 2) We then show that the two chains converge to each other, i.e., they converge to a single chain, called *right-left chain*. This construction does not directly guarantee that the right-left chain is a mutual chain. We then show a contradiction, reasoning on the total length of the segment $[R_0, R_n]$. 3) A consequence of the right-left chain not being a mutual chain is that the total length of the segment between R_0 and R_n is strictly smaller than $(j+1)V$ (where $j+1$ is the number of intervals between consecutive robots in the chain). 4) On the other hand, each such interval converges to V, thus implying that the total length of the segment is a number arbitrarily close to $(j+1)V$. This contradiction implies that the right-left chain is indeed mutual.

1) Construction of forward and backward chains. Let us consider a configuration of robots $\{x_0(t), x_1(t), \ldots, x_{j+1}(t)\} \subseteq \mathcal{R}(t)$, called *forward chain* (refer to Fig. 5), such that:

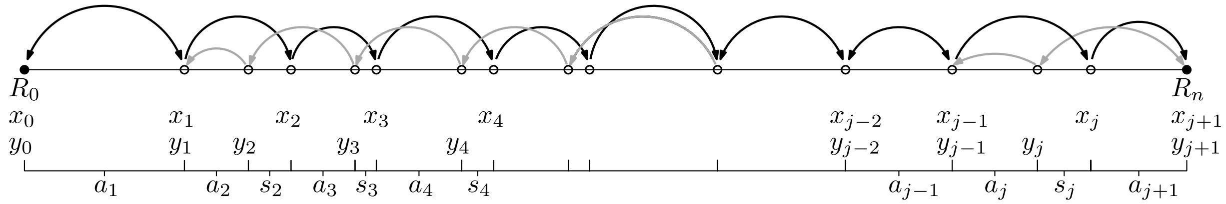

Fig. 5. Illustration of the proof of Theorem 4.9.

- $x_0(t) = x_0 = R_0$,
- $x_{i+1}(t) = r(x_i(t))$ for all $0 \le i \le j < n$
- and $x_{j+1}(t) = x_{j+1} = R_n$

We define another configuration of robots, called *backward chain*, $\{y_0(t), y_1(t), \ldots, y_{j+1}(t)\} \subseteq \mathcal{R}(t)$ as follows. Let $y_{j+1}(t) = x_{j+1}(t)$ and for all $0 \le i \le j$, let $y_i(t) = l(y_{i+1}(t))$ (refer to Fig. 5). Let us call the union of the two chains *right-left chain*. We now prove two useful properties about the right-left chain.

Property 1 (Alternation property.) For all $1 \le i \le j+1$, we have $x_{i-1}(t) < y_i(t) \le x_i(t)$. We prove this property by induction, starting at $i = j+1$. For the base case, notice that $y_{j+1}(t) = x_{j+1}(t)$ by definition. Suppose that $x_{i-1}(t) < y_i(t) \le x_i(t)$ for some $1 \le i \le j+1$. Then, $y_{i-1}(t) = l(y_i(t)) \le x_{i-1}(t)$ otherwise this would contradict the fact that $r(x_{i-1}(t)) = x_i(t)$. Moreover, $x_{i-2}(t) < l(y_i(t)) = y_{i-1}(t)$ otherwise this would contradict the fact that $r(x_{i-2}(t)) = x_{i-1}(t)$.

Property 2 (Starting point property.) We have that $y_0(t) = y_0 = R_0$. Indeed,

$$\begin{aligned} y_0(t) &= l(y_1(t)) && \text{by the definition of } y_0(t), \\ &\le l(x_1(t)) && \text{by Property 1,} \\ &= R_0, \end{aligned}$$

otherwise $x_1(t)$ would not be the rightmost visible robot from $R_0 = x_0$.

2) Convergence of forward and backward chains to a right-left chain. Notice that since the forward chain $\{x_0(t), x_1(t), \ldots, x_{j+1}(t)\}$ is not a mutual chain, there exists an i with $1 \le i \le j$ such that $x_{i-1}(t) < y_i(t) < x_i(t)$. For all $1 \le i \le j+1$, let $a_i(t) = y_i(t) - x_{i-1}(t)$ and $s_i(t) = x_i(t) - y_i(t)$. In what follows, our aim is to prove that $x_i(t)$ and $y_i(t)$ get arbitrarily close whenever $t \to \infty$. We do this by showing that $s_i(t) \xrightarrow{t\to\infty} 0$.

From Property 1, we have $a_i(t) > 0$ and $s_i(t) \ge 0$ for all $1 \le i \le j+1$. Moreover, $s_i(t) = 0$ if and only if $y_i(t) = x_i(t)$. Notice that $l(x_i(t-1)) \le x_{i-1}(t-1)$, otherwise there would be a contradiction with the fact that $r(x_{i-1}(t-1)) = x_i(t-1)$. Therefore,

$$x_i(t) = \frac{l(x_i(t-1)) + r(x_i(t-1))}{2} \le \frac{x_{i-1}(t-1) + x_{i+1}(t-1)}{2},$$

from which

$$x_i(t) \le \begin{cases} R_0 & i = 0, \\ x_{i-1}(t-1) + \frac{1}{2}(a_i(t-1) + s_i(t-1) + a_{i+1}(t-1) + s_{i+1}(t-1)) & 1 \le i \le j, \\ R_n & i = j+1. \end{cases} \tag{16}$$

Moreover, notice that $r(y_i(t-1)) \ge y_{i+1}(t-1)$, otherwise there would be a contradiction with the fact that $l(y_{i+1}(t-1)) = y_i(t-1)$. Therefore,

$$y_i(t) = \frac{l(y_i(t-1)) + r(y_i(t-1))}{2} \ge \frac{y_{i-1}(t-1) + y_{i+1}(t-1)}{2},$$

from which

$$y_i(t) \ge \begin{cases} R_0 & i = 0, \\ y_{i-1}(t-1) + \frac{1}{2}(s_{i-1}(t-1) + a_i(t-1) + s_i(t-1) + a_{i+1}(t-1)) & 1 \le i \le j, \\ R_n & i = j+1. \end{cases} \tag{17}$$

Since $s_i(t) = x_i(t) - y_i(t)$, by subtracting (17) from (16) we obtain

$$s_i(t) \leq \begin{cases} 0 & i = 0, \\ \frac{1}{2}(s_{i-1}(t-1) + s_{i+1}(t-1)) & 1 \leq i \leq j, \\ 0 & i = j+1. \end{cases} \tag{18}$$

We are now ready to prove that for all $0 \leq i \leq j+1$, $s_i(t) \xrightarrow{t\to\infty} 0$, implying that $|x_i(t) - y_i(t)| \xrightarrow{t\to\infty} 0$. Notice that we already have $y_0(t) = x_0(t)$ and $y_{j+1}(t) = x_{j+1}(t)$ by definition. We then have:

$$\begin{aligned} s_i(t) &\leq \frac{1}{2}(s_{i-1}(t-1) + s_{i+1}(t-1)) \\ &\leq \frac{1}{4}(s_{i-2}(t-2) + 2s_i(t-2) + s_{i+2}(t-2)) \\ &\leq \frac{1}{8}(s_{i-3}(t-3) + 3s_{i-1}(t-3) + 3s_{i+1}(t-3) + s_{i+3}(t-3)) \\ &\leq \frac{1}{16}(s_{i-4}(t-4) + 4s_{i-2}(t-4) + 6s_i(t-4) + 4s_{i+2}(t-4) + s_{i+4}(t-4)) \\ &\vdots \\ &\leq \frac{1}{2^t}\sum_{k=0}^{t}\binom{t}{k}s_{i-t+2k}(0), \end{aligned}$$

where $s_i(t) = 0$ for all $i \leq 0$ and $i \geq j+1$.

In order to determine the limit of $s_i(t)$ when $t \to \infty$, we need to make a few observations. First of all, the $s_i(t)$'s in the summation with $i \leq 0$ or $i \geq j+1$ are all equal to zero. In other words, regardless of the value of t, there are at most j non-zero values in the summation. Also note that since the segment delimited by the two faulty robots has a constant size, the values of the s_i's are bounded. Let C be the value of the largest such s_i ever occurring. Since the largest binomial coefficient is the central one (or the central ones for odd values of t), we can write

$$0 \leq s_i(t) \leq \frac{1}{2^t}\, j \binom{t}{\lfloor \frac{t}{2} \rfloor} C.$$

Since[4] $\binom{t}{\lfloor \frac{t}{2} \rfloor} \sim \frac{2^t}{\sqrt{\pi \frac{t}{2}}}$, we have

$$0 \leq \lim_{t\to\infty} s_i(t) \leq \lim_{t\to\infty} \frac{1}{2^t}\, j \binom{t}{\lfloor \frac{t}{2} \rfloor} C = \lim_{t\to\infty} \frac{1}{2^t}\, j \frac{2^t}{\sqrt{\pi \frac{t}{2}}} C = 0,$$

from which $\lim_{t\to\infty} s_i(t) = 0$.

We are ready to derive a contradiction.

3) Length of the segment strictly smaller than $(j+1)V$**.** Since the right-left chain is not a mutual chain, and x_0 and x_n are not moving, the distance between x_0 and x_n must be strictly smaller than $(j+1)V$ (otherwise x_j and y_j would necessarily coincide, for all j). So, there exists a real number $\delta > 0$ such that $x_n - x_0 = R_n - R_0 = (j+1)V - \delta$.

4) Distance between $x_i(t)$ **and** $x_{i+1}(t)$ **tending to** V**.** Let us consider any sub-chain of the right-left chain for which the x_i's and the y_i's are distinct except for the extremal ones. More precisely, let α and β be two indices such that $x_\alpha = y_\alpha$, $x_\beta = y_\beta$ and $x_i \neq y_i$ for all $\alpha < i < \beta$ (refer to Fig. 6). Notice that $l(x_{\alpha+1}) = x_\alpha = y_\alpha$, otherwise this would contradict the fact that $l(y_{\alpha+1}) = x_\alpha = y_\alpha$. We also have $r(y_{\beta-1}) = x_\beta = y_\beta$, otherwise this would contradict the fact that $r(x_{\beta-1}) = x_\beta = y_\beta$. Therefore,

$$\begin{aligned} l(x_{\alpha+1}) &= x_\alpha = y_\alpha, \\ r(x_{\alpha+1}) &= x_{\alpha+2}, \\ l(y_{\beta-1}) &= y_{\beta-2}, \\ r(y_{\beta-1}) &= x_\beta = y_\beta. \end{aligned}$$

[4] We write $f(t) \sim g(t)$ whenever $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

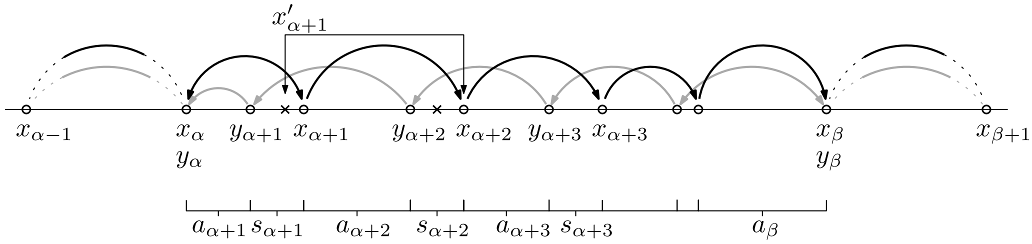

Fig. 6. Illustration of the contradiction in the proof of Theorem 4.9. We do not make any assumption about $x_{\alpha-1}$ being equal or not to $y_{\alpha-1}$, nor about $x_{\beta+1}$ being equal or not to $y_{\beta+1}$.

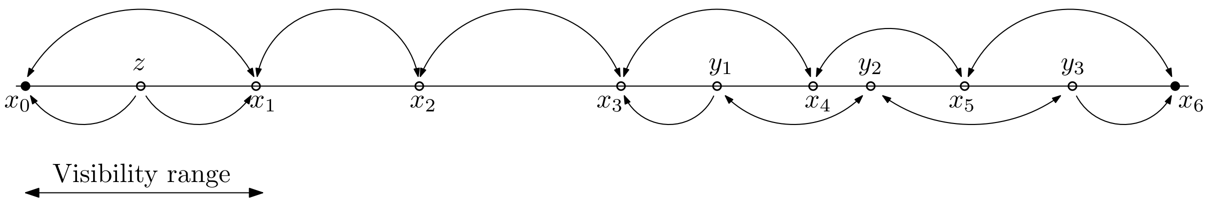

Fig. 7. An example of a primary chain $\{x_0, x_1, \ldots x_6\}$ with two level-2 chains: $\{z\}$ (anchored at x_0 and x_1) and $\{y_1, y_2, y_3\}$ (anchored at x_3 and x_6).

Notice that $x_{\alpha+1}$ and $y_{\beta-1}$ cannot have the same leftmost and rightmost visible robots, otherwise they would merge in one step, which is not possible at a size-stable time. This implies that $\beta > \alpha + 2$ (we need this to apply Lemma 4.10 later on). Since there cannot be any merging, given that $l(y_{\alpha+1}) = x_\alpha = y_\alpha$, we must also have that $x_{\alpha+2}$ is not visible from $y_{\alpha+1}$ at any time. Therefore, for all $t \geq 0$, $s_{\alpha+1}(t) + a_{\alpha+2}(t) + s_{\alpha+2}(t) > V$. Since $r(x_{\alpha+1}) = x_{\alpha+2}$, for all $t \geq 0$, $a_{\alpha+2}(t) + s_{\alpha+2}(t) \leq V$. Together with the fact that $s_{\alpha+1}(t) \overset{t\to\infty}{\longrightarrow} 0$ and $s_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} 0$, we get that $a_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$. Therefore, $x_{\alpha+2}(t) - x_{\alpha+1}(t) \overset{t\to\infty}{\longrightarrow} V$.

Our goal is to apply Lemma 4.10 and conclude that $x_{\alpha+1}(t) - x_\alpha(t) \overset{t\to\infty}{\longrightarrow} V$ and $x_{\alpha+3}(t) - x_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$. However, since $x_{\alpha+1}(t)$ and $x_{\alpha+2}(t)$ are not mutual, we cannot apply Lemma 4.10 directly. Let $x'_{\alpha+1}(t) = l(x_{\alpha+2}(t))$. Notice that we must have the following two inequalities:

$$y_{\alpha+1}(t) \leq x'_{\alpha+1}(t) \leq x_{\alpha+1}(t),$$

otherwise we would have contradictions, respectively, with the following two facts:

$$l(y_{\alpha+2}(t)) = y_{\alpha+1}(t), \quad r(x_{\alpha+1}(t)) = x_{\alpha+2}(t).$$

Moreover, we have $r(x'_{\alpha+1}(t)) = x_{\alpha+2}(t)$, otherwise this would contradict the fact that $r(x_{\alpha+1}(t)) = x_{\alpha+2}(t)$. Hence, $x'_{\alpha+1}(t)$ and $x_{\alpha+2}(t)$ are mutually chained (refer to Fig. 6). We now apply Lemma 4.10 on $x'_{\alpha+1}(t)$ and $x_{\alpha+2}(t)$.

Since $|x_{\alpha+1}(t) - y_{\alpha+1}(t)| \overset{t\to\infty}{\longrightarrow} 0$, then $|x_{\alpha+1}(t) - x'_{\alpha+1}(t)| \overset{t\to\infty}{\longrightarrow} 0$. The fact that $x_{\alpha+2}(t) - x_{\alpha+1}(t) \overset{t\to\infty}{\longrightarrow} V$ therefore implies that $x_{\alpha+2}(t) - x'_{\alpha+1}(t) \overset{t\to\infty}{\longrightarrow} V$. By Lemma 4.10, $x'_{\alpha+1}(t) - l(x'_{\alpha+1}(t)) \overset{t\to\infty}{\longrightarrow} V$ and $r(x_{\alpha+2}(t)) - x_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$. We have $l(x'_{\alpha+1}(t)) = x_\alpha(t) = y_\alpha(t)$, otherwise this would contradict the fact that $l(y_{\alpha+1}(t)) = x_\alpha(t) = y_\alpha(t)$. Therefore, the fact that $x'_{\alpha+1}(t) \in [y_{\alpha+1}(t), x_{\alpha+1}(t)]$, together with the fact that $|x_{\alpha+1}(t) - y_{\alpha+1}(t)| \overset{t\to\infty}{\longrightarrow} 0$, imply that $x_{\alpha+1}(t) - x_\alpha(t) \overset{t\to\infty}{\longrightarrow} V$. Moreover, since $r(x_{\alpha+2}(t)) = x_{\alpha+3}(t)$, the fact that $r(x_{\alpha+2}(t)) - x_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$ implies that $x_{\alpha+3}(t) - x_{\alpha+2}(t) \overset{t\to\infty}{\longrightarrow} V$.

By the previous argument, the fact that $x_{\alpha+2}(t) - x_{\alpha+1}(t) \overset{t\to\infty}{\longrightarrow} V$ propagates to $x_{\alpha+1}(t) - x_\alpha(t)$ and $x_{\alpha+3}(t) - x_{\alpha+2}(t)$. We can repeat the same argument and show that this propagates to all x_i's, from which we get that for all $0 \leq i \leq j$, $x_{i+1}(t) - x_i(t) \overset{t\to\infty}{\longrightarrow} V$. Therefore, the total distance between $x_0 = R_0$ and $x_n = R_n$ is arbitrarily close to $(j+1)V$. This contradicts the fact that $x_n - x_0 = R_n - R_0 = (j+1)V - \delta$ for all $t \geq 0$. □

In the proof of Theorem 4.9, we showed the existence of a unique mutual chain called the primary chain. Intuitively, we say that a configuration of robots is a secondary chain if it is a mutual chain anchored at two robots that belong to the primary chain. However, such a configuration is not necessarily unique (refer to Fig. 7 for an example). Level-j chains (for $j > 2$) are defined in a similar way.

Definition 4 *(Secondary chains and level-j chains).*

- The primary chain is called a *level-1 chain*.
- A configuration of robots C is a *secondary chain* if it is a mutual chain anchored at two robots x and y, such that $x, y \in \mathcal{C}_1$ and least one of x and y is non-faulty. We say that a secondary chain is a *level-2 chain*.

- A configuration of robots C is a *level-j chain* if it is a mutual chain anchored at two robots x and y which satisfy the following property. There exists an index $j' < j$ such that one of the following two statements is true:
  - x is part of a level-j' chain and y is part of a level-$(j-1)$ chain
  - or x is part of a level-$(j-1)$ chain and y is part of a level-j' chain.

The convergence of the primary chain can be proven by observing that the behaviour of the robots in the primary chain executing our algorithm (Convergence1D) is equivalent to the behaviour they would have if they were executing Algorithm Spreading. Once this is established, convergence follows from Theorem 3.2. The following lemma shows under what conditions Theorem 3.2 can be applied to a general mutual chain $Y(t) = \{y_1(t), y_2(t), \ldots, y_k(t)\}$. More specifically, suppose that there exist two real numbers y'_0 and y'_{k+1} such that $y_0(t) = l(y_1(t)) \overset{t\to\infty}{\longrightarrow} y'_0$ and $y_{k+1}(t) = r(y_k(t)) \overset{t\to\infty}{\longrightarrow} y'_{k+1}$. Then, by applying Algorithm Convergence1D, $Y(t)$ converges towards an equidistant configuration between y'_0 and y'_{k+1}.

Lemma 4.11. *Let $Y(t) = \{y_1(t), y_2(t), \ldots, y_k(t)\}$ be a mutual chain at a size-stable time t, anchored in $y_0(t) = l(y_1(t))$ and $y_{k+1}(t) = r(y_k(t))$, where $y_0(t) \neq y_1(t)$ and $y_{k+1}(t) \neq y_k(t)$. Suppose that there exist two numbers y'_0 and y'_{k+1}, such that $y_0(t) \overset{t\to\infty}{\longrightarrow} y'_0$ and $y_{k+1}(t) \overset{t\to\infty}{\longrightarrow} y'_{k+1}$. We have that, for all $0 \le i \le k+1$,*

$$y_i(t) \overset{t\to\infty}{\longrightarrow} y'_0 + \frac{|y'_{k+1} - y'_0|}{k+1} i.$$

Therefore, as $t \to \infty$, the robots in $\{y_1(t), y_2(t), \ldots, y_k(t)\}$ converge to a configuration where the distance between any two consecutive robots is $\frac{|y'_{k+1} - y'_0|}{k+1}$.

Proof. Let $Z(t) = \{z_0(t) = y_0(t), z_1(t), z_2(t), \ldots, z_m(t) = y_{k+1}(t)\} \subseteq \mathcal{R}(t)$ be the global configuration of robots at time t, restricted to the interval $[y_0(t), y_{k+1}(t)]$.

By Theorem 4.8, $Y(t)$ satisfies the following property: for all $1 \le i \le k$ and for all $t' \ge t$, we have that $l(y_i(t')) = l(y_i(t))$ and $r(y_i(t')) = r(y_i(t))$. Therefore, even if there is a robot $z_j(t) \in N(y_i(t)) \setminus Y(t)$, the presence of $z_j(t)$ has no impact on the position of $y_i(t+1)$. Consequently, the positions of the robots in $Y(t+1)$, after executing Algorithm Convergence1D on $Y(t)$, are uniquely determined by the positions of the robots in $Y(t)$. Hence, executing Algorithm Convergence1D on $Y(t)$ produces the same result as executing Algorithm Spreading on $Y(t)$, and thus the lemma follows from Theorem 3.2. □

We now show that the primary chain $\mathcal{C}_1 = \{x_0, x_1, x_2, \ldots, x_k\} \subseteq \mathcal{R}$, where $x_0 = R_0$ and $x_k = R_n$, converges towards a configuration of equidistant robots delimited by its anchors R_0 and R_n.

Theorem 4.12 *(Convergence of the primary chain). Let $\mathcal{C}_1 = \{x_0, x_1, x_2, \ldots, x_k\}$ be the primary chain. We have that $x_0 = R_0$, $x_k = R_n$ and for all $0 \le i \le k$*

$$x_i(t) \overset{t\to\infty}{\longrightarrow} \frac{|R_n - R_0|}{k} i.$$

Proof. Since $\mathcal{C}_1$ is a mutual chain, the configuration $\{x_1, x_2, \ldots, x_{k-1}\}$ is also a mutual chain. It is anchored at x_0 and x_k, where $x_0 \neq x_1$ and $x_k \neq x_{k-1}$. Since the anchors $x_0 = R_0 = 0$ and $x_k = R_n$ are faulty, they do not move. Hence, $x_0(t) \overset{t\to\infty}{\longrightarrow} R_0 = 0$ and $x_k(t) \overset{t\to\infty}{\longrightarrow} R_n$. Thus, the theorem follows directly from Lemma 4.11. □

We now show that every level-j chain converges towards a configuration of equidistant robots.

Theorem 4.13 *(Convergence of level-j chains). Let $C_j = \{y_1, y_2, \ldots, y_k\}$ be a level-j chain, where $j \ge 1$ is an integer. Let t be a size-stable time. Let $y_0(t) = l(y_1(t))$ and $y_{k+1}(t) = r(y_k(t))$. There exist real numbers y'_0 and y'_{k+1} such that $y_0(t) \overset{t\to\infty}{\longrightarrow} y'_0$ and $y_{k+1}(t) \overset{t\to\infty}{\longrightarrow} y'_{k+1}$. Moreover, for all $0 \le i \le k+1$,*

$$y_i(t) \overset{t\to\infty}{\longrightarrow} y'_0 + \frac{|y'_{k+1} - y'_0|}{k+1} i.$$

Proof. We proceed by induction on j. By Definition 4, a level-1 chain is a primary chain. Therefore, by Theorem 4.12, our statement is true for $j = 1$. Suppose that the theorem is true for all integers from 1 to $j-1$. Consider a level-j chain $C_j = \{y_1, y_2, \ldots, y_k\}$ anchored at $y_0(t) = l(y_1(t))$ and $y_{k+1}(t) = r(y_k(t))$, where t is a size-stable time.

By Definition 4, there exists an index $j' < j$ such that one of the following two statements is true:

- y_0 is part of a level-j' chain and y_{k+1} is part of a level-$(j-1)$ chain
- or y_0 is part of a level-$(j-1)$ chain and y_{k+1} is part of a level-j' chain.

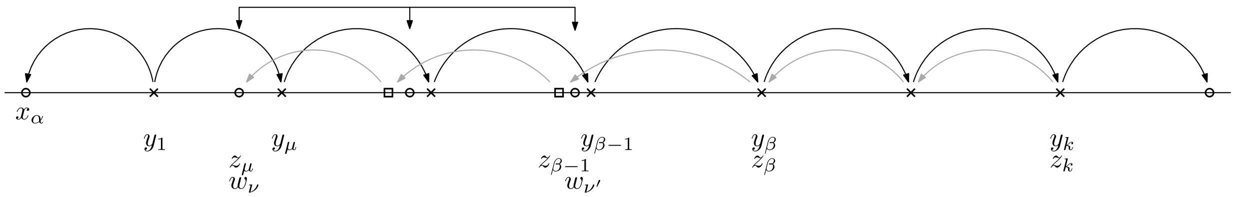

Fig. 8. Illustration of the proof of Lemma 4.14.

Without loss of generality, suppose that y_0 is part of a level-j' chain and y_{k+1} is part of a level-$(j-1)$ chain.

By the induction hypothesis, there exist two real numbers y'_0 and y'_{k+1} such that $y_0(t) \overset{t\to\infty}{\longrightarrow} y'_0$ and $y_{k+1}(t) \overset{t\to\infty}{\longrightarrow} y_{k+1}$. The theorem follows from Lemma 4.11. $\square$

The following lemma states that every robot belongs to some level-j chain. To simplify the presentation, we assume that the faulty robot x_0 is part of the *level*-0 chain $\{x_0\}$ and that the faulty robot x_n is part of the *level*-0 chain $\{x_n\}$.

Lemma 4.14. *For all size-stable time t and all $0 \le i \le n$, there exists an integer $j \ge 0$ such that $R_i(t) \in \mathcal{R}(t)$ belongs to a level-j chain.*

Proof. Suppose that the statement is false. Let $y_1(t)$ be the leftmost robot that does not belong to any level-j chain. Therefore, $l(y_1(t))$ belongs to a level-j' chain (for some $j' \ge 0$), say $C(t) = \{x_1, x_2, ..., x_m\}$, where $l(y_1(t)) = x_\alpha(t)$ for some index $1 \le \alpha \le m$. Since $y_1(t)$ does not belong to any level-j chain, $r(y_1(t))$ does not belong to any level-j chain. Otherwise, by definition, $\{y_1(t)\}$ would be a level-j'' chain for some $j'' \ge j'$. Therefore, assume that $r(y_1(t))$ does not belong to any level-j chain.

Let $Y = \{y_1, y_2, ..., y_k\}$ be the configuration of robots such that (refer to Fig. 8)

1. $y_i(t) = r(y_{i-1}(t))$, where $2 \le i \le k$,
2. for all $1 \le i \le k$, $y_i(t)$ does not belong to any level-j chain
3. and $r(y_k(t))$ belongs to a level-j'' chain (for some $j'' \ge j'$).

Observe that Y is well-defined. Indeed, by the previous discussion, $k \ge 2$. Moreover, by construction, $\{y_1(t), y_2(t), ..., y_k(t)\}$ is not a mutual chain.

Let $\{z_1, z_2, ..., z_k\}$ be the configuration of robots such that $z_k = y_k$ and $z_i(t) = l(z_{i+1}(t))$ for all $1 \le i \le k-1$. Using the same arguments as in the proof of Theorem 4.9, we get that $x_\alpha \le z_1 \le y_1$ and $y_{i-1} < z_i \le y_i$ for all $2 \le i \le k$. Since $\{y_1(t), y_2(t), ..., y_k(t)\}$ is not a level-j chain for any $j \ge 0$, there is an index $1 \le i \le k$ such that $z_i(t) \ne y_i(t)$. Let β be the smallest index such that $z_\beta = y_\beta$ and $z_{\beta-1} \ne y_{\beta-1}$.

We claim that $z_i(t) \ne y_i(t)$ for all $1 \le i < \beta - 1$. We prove this claim by contradiction. Assume there is an index $\gamma < \beta - 1$ such that $z_\gamma(t) = y_\gamma(t)$. Therefore, by the definition of β, $z_i = y_i$ for all $1 \le i \le \gamma$. Moreover, $x_\alpha = l(y_1)$ and $r(y_k)$ are part of level-j chains. Therefore, by Theorems 4.12 and 4.13, $x_\alpha(t) = l(y_1(t))$ and $r(y_k)(t)$ converge to a fixed location as $t \to \infty$. Consequently, when we consider the configurations $\{y_1, y_2, ..., y_k\}$ and $\{z_1, z_2, ..., z_k\}$, we get the same contradiction as in the proof of Theorem 4.9. This proves our claim.

We have the following property.

Property 1 If, for all $2 \le i \le \beta - 1$, $z_i(t)$ does not belong to any level-j chain, then $z_1(t) = l(z_2(t))$ belongs to a level-j chain. Indeed, we must have $z_1(t) \le y_1(t)$ otherwise this would contradict the fact that $y_2(t) = r(y_1(t))$. Moreover, we assumed that $z_i(t) \ne y_i(t)$ for all $1 \le i < \beta - 1$. Hence, $z_1(t) < y_1(t)$. Moreover, we must have $z_1(t) \ge x_\alpha(t)$ otherwise this would contradict the fact that $x_\alpha(t) = l(y_1(t))$. But then, since $y_1(t)$ is the leftmost robot that does not belong to any level-j chain, we must have that $z_1(t) = l(z_2(t))$ belongs to a level-j chain.

Consequently, there is an index $1 \le i \le \beta - 1$ such that $z_i(t)$ belongs to a level-j chain. Let $1 \le \mu \le \beta - 1$ be the largest index such that $z_\mu(t)$ belongs to a level-j chain, say $W = \{w_1, w_2, ..., w_{m'}\}$. Let $1 \le \nu \le m'$ be the index such that $w_\nu(t) = z_\mu(t)$.

We have the following property.

Property 2 $z_{\mu+1}(t) < w_{\nu+1}(t) < y_{\mu+1}(t)$. Indeed, observe that $w_{\nu+1}(t) = r(w_\nu(t))$. Therefore, $w_{\nu+1}(t) \le y_{\mu+1}(t)$, otherwise this would contradict the fact that $y_{\mu+1}(t) = r(y_\mu(t))$. Moreover, by definition, $w_{\nu+1}(t) \ne y_{\mu+1}(t)$. We also have that $w_{\nu+1}(t) \ge z_{\mu+1}(t)$ otherwise this would contradict the fact that $z_\mu(t) = l(z_{\mu+1}(t))$. Moreover, by definition, $w_{\nu+1}(t) \ne z_{\mu+1}(t)$.

By repeating the argument for proving Property 2, we reach the index ν' such that $z_{\beta-1}(t) < w_{\nu'}(t) < y_{\beta-1}(t)$. Observe that $w_{\nu'+1}(t) = r(w_{\nu'}(t)) \le y_\beta(t)$, otherwise this would contradict the fact $y_\beta(t) = r(y_{\beta-1}(t))$. Moreover, $w_{\nu'+1}(t) \ge y_\beta(t) =$

$z_\beta(t)$, otherwise this would contradict the fact $z_{\beta-1}(t) = l(z_\beta(t))$. Therefore, $w_{\nu'+1}(t) = y_\beta(t)$. However, by the definition of Y, $y_\beta(t)$ is not part of any level-j chain. We get a contradiction. □

The following theorem follows directly from Theorems 4.12 and 4.13, and Lemma 4.14.

Theorem 4.15 *(Global convergence). For all $0 \le i \le n$, there exists a real number R_i^* such that $R_i(t) \xrightarrow{t\to\infty} R_i^*$. Therefore, $\mathcal{R}(t)$ converges towards a fixed configuration $\mathcal{R}^* = \{R_0^*, R_1^*, \ldots, R_n^*\}$ as $t \to \infty$. The configuration $\mathcal{R}^*$ contains a primary chain $\mathcal{C}_1$ anchored at R_0 and R_n. Additionally, there is an integer $\kappa \ge 1$ such that for all $0 \le i \le n$, R_i^* belongs to a level-j chain, for some $1 \le j \le \kappa$. Moreover, every level-j chain in $\mathcal{R}^*$ is a mutual chain of equidistant robots.*

5. Conclusion

To study the impact of faults on the robots dynamics, in this paper we analyzed the behaviour of a group of oblivious robots which execute an algorithm designed for a fault-free environment in presence of undetectable crash faults. We focused on the classic point-convergence algorithm by Ando et al. [2] executed on a line, when the robots are synchronous and at most two of them are faulty.

The paper leaves several open questions and research directions.

An obvious extension would be the study of the point-convergence algorithm in the case of *more than two faults*. Extensive simulation indicates that the robots still converge to some more complex combination of mutual chains. However, several situations can occur depending on the location of the faulty robots, on their relative distance, as well as on their number. The characterization of the family of fixed points is far from simple and it is left for further study.

The study of this system under a *semi-synchronous scheduler* (SSYNC) would be quite interesting. In SSYNC, an arbitrary subset of the robots is activated at each round. The choice is made by an adversary, which has only to insure that every robot is activated infinitely often (i.e., that no robot is left inactive for ever after any given time). The analysis of the long term behaviour of the system under SSYNC is quite complicated and it is not even clear whether the system converges or not. One of the challenges is that the inactivity of a robot behaves as a fault, for an arbitrary amount of time, and a clever activation pattern chosen by the adversary might make the system oscillate between different configurations. A study of this setting is left as future work.

Finally, we know that when the robots operate fully synchronously in a *two dimensional space* (i.e., in the plane), their dynamics has a rather different nature. In fact, they seem to converge in most cases, but it can be shown that some executions lead to an oscillating behaviour. The study of this case is undergoing.

More generally, this work can be seen as a first step toward the study of the interaction between heterogeneous groups of robots operating in the same space, each following a different algorithm. The existing literature on LOOK-COMPUTE-MOVE robots has always considered robots with the same set of rules. The presence of different teams following different, possibly conflicting, rules in the environment is an interesting new area of investigation.

Declaration of competing interest

There is no competing interest.

Appendix A. Proof of Lemma 3.3

In this section, we provide a complete and detailed proof of Lemma 3.3.

Proof. By the hypothesis, the lemma is true for $i = 0$ and $i = m$. If $m = 2$, by the hypothesis,

$$|\eta_1(t+1)| = \left|\frac{\eta_0(t) + \eta_2(t)}{2}\right| \le \frac{|\eta_0(t)| + |\eta_2(t)|}{2} \xrightarrow{t\to\infty} 0.$$

Hence, assume that $m \ge 3$. To deal with other values of i, let

$$\psi(t) = \sum_{i=0}^{m} \eta_i^2(t).$$

We show that $\psi(t) \xrightarrow{t\to\infty} 0$, which completes the proof. Following the same approach as the one used in the proof of Theorem 3.1, we use the Fourier sine series of $\eta_i(t)$. However, in our case, we need to be careful since $\eta_0(t)$ and $\eta_m(t)$ are not necessarily equal to 0. For all $0 \le i \le m$, $0 \le k \le m$ and $t \ge 0$, let

$$g(i,k) = \sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \qquad \text{and} \qquad \mu_k(t) = \sum_{i=0}^{m} \eta_i(t) g(i,k).$$

We have

$$\eta_i(t) = \sum_{k=0}^{m} \mu_k(t) g(i,k) \tag{A.1}$$

for all $1 \le i \le m-1$ and $t \ge 0$. Moreover, we have

$$g(i,0) = g(i,m) = g(0,k) = g(m,k) = 0 \tag{A.2}$$

for all $0 \le i \le m$ and $0 \le k \le m$, from which

$$\sum_{k=0}^{m} \mu_k(t) g(0,k) = \sum_{k=0}^{m} \mu_k(t) g(m,k) = 0.$$

Observe that for all $0 \le k \le m$ and $0 \le q \le m$,

$$\sum_{i=0}^{m} g(i,k) g(i,q) = \begin{cases} 0 & \text{if } k=0, k=m, q=0 \text{ or } q=m \\ \delta_{k,q} & \text{otherwise,} \end{cases} \tag{A.3}$$

where $\delta_{k,q}$ stands for the Kronecker's delta, i.e., $\delta_{k,q} = 1$ if $k = q$ and 0 otherwise. Moreover, observe that for all $0 \le k \le m$ and $t \ge 0$,

$$|\mu_k(t)| = \left| \sum_{i=0}^{m} \eta_i(t) g(i,k) \right| = \left| \sum_{i=1}^{m-1} \eta_i(t) g(i,k) \right| \le \sum_{i=1}^{m-1} |\eta_i(t) g(i,k)| \le (m-1)\sqrt{\frac{2}{m}} M \le \sqrt{2m} M. \tag{A.4}$$

We also have

$$\begin{aligned}
\psi(t) &= \sum_{i=0}^{m} \eta_i^2(t) \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{i=1}^{m-1} \eta_i^2(t) \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{i=1}^{m-1} \left(\sum_{k=0}^{m} \mu_k(t) g(i,k) \right)^2 && \text{by (A.1),} \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{i=0}^{m} \left(\sum_{k=0}^{m} \mu_k(t) g(i,k) \right)^2 && \text{by (A.2),} \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{i=0}^{m} \left(\sum_{k=0}^{m} \sqrt{\frac{2}{m}} \mu_k(t) \sin\left(\frac{ki\pi}{m}\right) \right)^2 \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{i=0}^{m} \sum_{k=0}^{m} \sum_{q=0}^{m} \left(\sqrt{\frac{2}{m}} \mu_k(t) \sin\left(\frac{ki\pi}{m}\right) \right) \left(\sqrt{\frac{2}{m}} \mu_q(t) \sin\left(\frac{qi\pi}{m}\right) \right) \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{k=0}^{m} \sum_{q=0}^{m} \mu_k(t) \mu_q(t) \sum_{i=0}^{m} \left(\sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \right) \left(\sqrt{\frac{2}{m}} \sin\left(\frac{qi\pi}{m}\right) \right) \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{k=0}^{m} \sum_{q=0}^{m} \mu_k(t) \mu_q(t) \sum_{i=0}^{m} g(i,k) g(i,q) \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{k=0}^{m} \sum_{q=0}^{m} \mu_k(t) \mu_q(t) \delta_{k,q} && \text{by (A.3) and since } \mu_0(t) = \mu_m(t) = 0, \\
&= \eta_0^2(t) + \eta_m^2(t) + \sum_{k=1}^{m-1} \mu_k^2(t) && \text{since } \mu_0(t) = \mu_m(t) = 0.
\end{aligned} \tag{A.5}$$

Moreover, for $1 \le k \le m-1$ (since $\mu_0(t+1) = \mu_m(t+1) = 0$), we have

$$\begin{aligned}
&\mu_k(t+1) \\
&= \sum_{i=0}^{m} \sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \eta_i(t+1) \\
&= \sum_{i=1}^{m-1} \sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \eta_i(t+1) \\
&= \sum_{i=1}^{m-1} \sqrt{\frac{2}{m}} \sin\left(\frac{ki\pi}{m}\right) \frac{\eta_{i-1}(t) + \eta_{i+1}(t)}{2} \\
&= \sum_{i=0}^{m-2} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i+1)\pi}{m}\right) \eta_i(t) + \sum_{i=2}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i-1)\pi}{m}\right) \eta_i(t) \\
&= \sum_{i=0}^{m-1} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i+1)\pi}{m}\right) \eta_i(t) + \sum_{i=1}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i-1)\pi}{m}\right) \eta_i(t) \\
&= -\frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(m+1)\pi}{m}\right) \eta_m(t) + \sum_{i=0}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i+1)\pi}{m}\right) \eta_i(t) \\
&\qquad - \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{-k\pi}{m}\right) \eta_0(t) + \sum_{i=0}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i-1)\pi}{m}\right) \eta_i(t) \\
&= \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k\pi}{m}\right) \eta_m(t) + \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k\pi}{m}\right) \eta_0(t) \\
&\qquad + \sum_{i=0}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i+1)\pi}{m}\right) \eta_i(t) + \sum_{i=0}^{m} \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k(i-1)\pi}{m}\right) \eta_i(t) \\
&= \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k\pi}{m}\right) (\eta_0(t) + \eta_m(t)) + \frac{1}{2}\sqrt{\frac{2}{m}} \sum_{i=0}^{m} \left(\sin\left(\frac{k(i+1)\pi}{m}\right) + \sin\left(\frac{k(i-1)\pi}{m}\right)\right) \eta_i(t) \\
&= \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k\pi}{m}\right) (\eta_0(t) + \eta_m(t)) + \sqrt{\frac{2}{m}} \sum_{i=0}^{m} \sin\left(\frac{ki\pi}{m}\right) \cos\left(\frac{k\pi}{m}\right) \eta_i(t) \\
&= \frac{1}{2}\sqrt{\frac{2}{m}} \sin\left(\frac{k\pi}{m}\right) (\eta_0(t) + \eta_m(t)) + \cos\left(\frac{k\pi}{m}\right) \mu_k(t) \\
&\le |\eta_0(t)| + |\eta_m(t)| + \cos\left(\frac{k\pi}{m}\right) \mu_k(t)
\end{aligned} \tag{A.6}$$

Therefore, since $\mu_0(t+1) = \mu_m(t+1) = 0$,

$$\begin{aligned}
&\psi(t+1) \\
&= \eta_0^2(t+1) + \eta_m^2(t+1) + \sum_{k=0}^{m} \mu_k^2(t+1) \\
&= \eta_0^2(t+1) + \eta_m^2(t+1) + \sum_{k=1}^{m-1} \mu_k^2(t+1) \\
&\le \eta_0^2(t+1) + \eta_m^2(t+1) + \sum_{k=1}^{m-1} \left(|\eta_0(t)| + |\eta_m(t)| + \cos\left(\frac{k\pi}{m}\right) \mu_k(t)\right)^2 \qquad \text{by (A.6)}, \\
&= \eta_0^2(t+1) + \eta_m^2(t+1) \\
&\qquad + \sum_{k=1}^{m-1} \left((|\eta_0(t)| + |\eta_m(t)|)^2 + 2\,(|\eta_0(t)| + |\eta_m(t)|) \cos\left(\frac{k\pi}{m}\right) \mu_k(t) + \cos^2\left(\frac{k\pi}{m}\right) \mu_k^2(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \eta_0^2(t+1) + \eta_m^2(t+1) + \sum_{k=1}^{m-1} (|\eta_0(t)| + |\eta_m(t)|)^2 \\
&\qquad + \sum_{k=1}^{m-1} 2\,(|\eta_0(t)| + |\eta_m(t)|) \cos\left(\frac{k\pi}{m}\right) \mu_k(t) + \sum_{k=1}^{m-1} \cos^2\left(\frac{k\pi}{m}\right) \mu_k^2(t) \\
&\le \eta_0^2(t+1) + \eta_m^2(t+1) + (m-1)\,(|\eta_0(t)| + |\eta_m(t)|)^2 \\
&\qquad + 2\,(|\eta_0(t)| + |\eta_m(t)|) \cos\left(\frac{\pi}{m}\right) \sum_{k=1}^{m-1} \mu_k(t) + \cos^2\left(\frac{\pi}{m}\right) \sum_{k=1}^{m-1} \mu_k^2(t) \\
&\le \eta_0^2(t+1) + \eta_m^2(t+1) + (m-1)\,(|\eta_0(t)| + |\eta_m(t)|)^2 + 2\,(|\eta_0(t)| + |\eta_m(t)|) \cos\left(\frac{\pi}{m}\right) (m-1)\sqrt{2m}M \\
&\qquad + \cos^2\left(\frac{\pi}{m}\right) \left(\psi(t) - \eta_0^2(t) - \eta_m^2(t)\right) \qquad \text{by (A.4) and (A.5),} \\
&\le \eta_0^2(t+1) + \eta_m^2(t+1) + (m-1)\,(|\eta_0(t)| + |\eta_m(t)|)^2 \\
&\qquad + 2\,(|\eta_0(t)| + |\eta_m(t)|)\,(m-1)\sqrt{2m}M + \cos^2\left(\frac{\pi}{m}\right) \psi(t). \qquad \text{(A.7)}
\end{aligned}
$$

Since $\eta_0(t) \stackrel{t\to\infty}{\longrightarrow} 0$ and $\eta_m(t) \stackrel{t\to\infty}{\longrightarrow} 0$, there exists a positive function $\epsilon(t)$ such that $\epsilon(t) \stackrel{t\to\infty}{\longrightarrow} 0$, $\epsilon(t) = O(1)$ and for all $t \ge 0$, we have $|\eta_0(t)| \le \epsilon(t)$, $|\eta_0(t+1)| \le \epsilon(t)$, $|\eta_m(t)| \le \epsilon(t)$ and $|\eta_m(t+1)| \le \epsilon(t)$. Therefore, the right-hand side of inequality (A.7) is upper bounded by

$$
\begin{aligned}
&\epsilon^2(t) + \epsilon^2(t) + 4(m-1)\epsilon^2(t) + 4\epsilon(t)(m-1)\sqrt{2m}M + \cos^2\left(\frac{\pi}{m}\right)\psi(t) \\
&\le \left(2\epsilon(t) + 4(m-1)\epsilon(t) + 4(m-1)\sqrt{2m}M\right)\epsilon(t) + \cos^2\left(\frac{\pi}{m}\right)\psi(t) \\
&\le \phi\,\epsilon(t) + \lambda\,\psi(t),
\end{aligned}
$$

where $\phi = 2\epsilon(t) + 4(m-1)\epsilon(t) + 4(m-1)\sqrt{2m}M = O(1)$ and $\lambda = \cos^2\left(\frac{\pi}{m}\right) \le \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$.

Consequently, by unfolding (A.7) t times, we get

$$
\begin{aligned}
\psi(t) &\le \left(1 + \lambda + \lambda^2 + \ldots + \lambda^{t-1}\right)\phi\,\epsilon(t) + \lambda^t\psi(0) \\
&= \frac{1-\lambda^t}{1-\lambda}\phi\,\epsilon(t) + \lambda^t\psi(0) \\
&\stackrel{t\to\infty}{\longrightarrow} 0. \quad \square
\end{aligned}
$$

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