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Cellular automata in fuzzy backgrounds ★

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Abstract

The main purpose of this work is to understand some limitations introduced by the classical definitions of cellular automata (CA). To this end, we have defined a new model of CAs (*fuzzy CAs*) which allows the observation of interesting "chaotic" properties of elementary CAs. To date neither a formal nor a precise definition of "chaos" in CAs exists; we believe that the proposed model provides a "sharper" tool to detect which properties can be associated to a "chaotic" behavior. We also define a measure (*rule entropy*) which gives information about the CA's dynamics solely on the basis of the rule table and provides theoretical explanations to some of the empirical observations.

Keywords: Cellular automata; Fuzzification; Classification

1. Introduction

Cellular automata (CA) are totally discrete dynamical systems. Discreteness of space implies a regular d-dimensional lattice with each site (the "cell" of the automata) labeled by a value (the "state" of the site) from a limited range of possible values. Discreteness of time means that the state of each site changes at successive steps by the iteration of a fixed CA rule, depending on the states of the "neighboring" sites. In other words, the new state of any cell (the "target" cell) at the instant $t+1$ is a function of the states and locations of a set of cells (the neighborhood) at t, typically situated locally in relation to the target cell. To be precise, the neighborhood of a site is defined as the site itself plus a certain number of adjacent sites. The pattern of states across the whole lattice is the CA " configuration" (or global state) at a given time. Any pattern may be set as an initial condition at time t. Each cell of the lattice simultaneously has its state updated and evolves to a new configuration at time $t+1$. Moreover, this process takes on synchronously for every site of the lattice [22].

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"Space–time patterns, which represent CA trajectories from given initial configurations, have been the focus of statistical analysis and classification, and have been extensively illustrated in the literature. Given the same CA architecture, different rules produce characteristic space–time patterns. [...] Space–time patterns, in very broad terms, are said to display behavior that is either static, periodic, complex (with interacting emergent structures), or *chaotic* [22]. CA rules' classification schemes have been made on the basis of such space–time pattern phenomenology". (from [23]).

In the literature, many attempts to classify CAs according to their asymptotic behavior are reported [3,5,9,11,18,19, 22], all based on the original classification of Wolfram [22] for one-dimensional CAs starting from finite configurations in quiescent background:

- *Class 1.* Automata that evolve to a unique, homogeneous state, after a finite transient (static CAs).
- *Class 2.* Automata whose evolution leads to a set of separated simple stable or periodic structures (space–time patterns) (periodic CAs).
- *Class 3.* Automata whose evolution leads to aperiodic ("chaotic") space–time patterns ("chaotic" CAs).
- *Class 4.* Automata that evolve to complex patterns with propagative localized structures, sometimes long-lived (complex CAs).

It has been shown that it is impossible to decide to which class a CA belongs [5]; actually it has been proved that it is undecidable even to determine whether a CA belongs to Class 1 [13].

Since class membership is undecidable, the observation of the evolution of a CA starting from (possibly all) initial configurations becomes crucial to understand its dynamics. The method employed by Wolfram to observe the evolution, and thus to propose an empirical assignment of CAs to classes, uses quiescent backgrounds and a constant-size window of observability (or circular CAs). These choices, motivated by the practical impossibility of working with the infinite, are in some cases impediments to the understanding of the inherent dynamics of some CAs. Two examples will clarify this point:

(1) In Wolfram's classification, the shift rule (170) is one of the simplest rules belonging to Class 1 (when observed in a quiescent background); in particular this rule, whose one-dimensional bi-infinite extension is the paradigm of chaos [1,6,21], is not considered chaotic. This is due to the fact that, within few steps, the quiescent background will fill the fixed-size window, and the only observable configuration is the quiescent one. Just moving to a nonquiescent background, the observation of a rather complex dynamics is possible [2].

(2) Another, perhaps more interesting, example is given by rules 90 and 18. Both rules are in Class 3, and are commonly used as the primary example of "chaotic" CAs [4,8,14,17,22]. Furthermore, all experimental observations, using quiescent background and fixed-size window, show no substantial difference in their behavior. Indeed, some suspicions that these two rules might have significantly different dynamics has been raised [10], but has never been experimentally observed.

In this work we aim to understand what are the limitations that the use of a zero-background and the fixed-size window has on the evolution of CAs, and thus, on the understanding of their dynamics. To this purpose, we define a new model (*CAs in fuzzy background*) where a finite window of boolean values is embedded in a background of rational values in [0,1] ("fuzzy" states), and the global function is a mapping between fuzzy configurations.

This new model overcomes the observed limitations of using a fixed-size window with quiescent background. In particular, it allows to detect complex dynamics of the shifting rules (e.g., 170). It also provides evidence that several apparently similar rules (e.g., rules 90 and 18, 57 and 184) indeed have a significantly different dynamics (see Figs. 4 and 5).

We also define a new measure (*rule entropy*) that gives information about the CA's dynamics solely on the basis of the rule table, and not on the basis of the dynamical evolution of the CAs. The rule entropy provides a theoretical explanation to some of the empirical observations. Furthermore, we show that the rule entropy captures the relationship between linearity of the rules and complex dynamics of the corresponding CAs.

Summarizing, this model allows for more accurate observations of the inherent dynamics of CAs by overcoming the problems related to the quiescent background and to the fixed-size window mentioned above.

The paper is organized as follows: in Section 2, we define the new model of CAs in fuzzy background, the fuzzy operators, the background, and the window of observability. In Section 3, we describe the empirical observations of the dynamics of CAs in fuzzy background and give our interpretations. In Section 4 rule entropy is defined and theoretical explanation of our interpretation is given. In Section 5, we use a notion of nonlinearity defined from the rules of CAs to link our previous results and observations with the chaotic behavior of CAs. Finally, we draw some conclusions in Section 6.

2. Cellular automata in fuzzy backgrounds

We define a new model (CAs in fuzzy background) where a finite window of boolean values is embedded in a background of rational values in [0,1] ("fuzzy" states).

In Section 2.1, we define the model; in Section 2.2, we describe the choice of fuzzy operators; in Section 2.3, we motivate the choice of the fuzzy background, introduce the region of observability and discuss the possible dynamics of CAs in the chosen setting.

2.1. Fuzzy cellular automata

Definition 1. A one-dimensional bi-infinite boolean CA is a quadruple

$$C_b = \langle Z, \{0, 1\}, r, f \rangle,$$

where Z is the set of cells; $i \in Z$ is the *location* of cell 'i'; $\{0, 1\}$ is the set of boolean *states* of the cells; $r \in \mathbb{N}$ is the *radius of the neighborhood*; $f : \{0, 1\}^{2r+1} \to \{0, 1\}$ is the *local function*, also called the *rule* of the automaton.

As a generalization of boolean CAs, we define fuzzy CAs.

Definition 2. A *one-dimensional, bi-infinite fuzzy CA* is a one-dimensional, bi-infinite CA

$$C_f = \langle Z, S, r, h \rangle,$$

where Z is the set of cells; $i \in Z$ is the *location* of cell 'i'; $S \subset [0, 1]$ is the finite set of rational *states* of the cells; $r \in \mathbb{N}$ is the *radius of the neighborhood*; $h : S^{2r+1} \to S$ is the *local function*, also called the *rule* of the automaton.

A *configuration* (or *global state*) of a fuzzy CA is a function $\underline{x} : Z \to S$ that specifies a state for each site of the lattice and can be represented by a bi-infinite sequence:

$$\underline{x} = (\ldots, x_{-m}, x_{-m+1}, \ldots, x_{-1}, x_0, x_{+1}, \ldots, x_{m-1}, x_m, \ldots).$$

The configuration space (also called phase space) of the CA is the set S^Z of all possible CA configurations. The *neighborhood* of a site $i \in Z$ is the set

$$\{i - r, i - r + 1, \ldots, i - 1, i, i + 1, \ldots, i + r - 1, i + r\},$$

that is, the r sites to the left and r sites to the right of i (plus i itself).

Table 1

	MV	Lukasievic	Probabilistic	Present paper
	$a \oplus b$	$a \sqcup b$	$a \hat{+} b$	$a \oplus b$
$\vee(a,b)$	$\min\{1, (a+b)\}$	$\max\{a,b\}$	$a+b-a\cdot b$	$\min\{1,(a+b)\}$
	$a \odot b$	$a \sqcap b$	$a \cdot b$	$a \cdot b$
$\wedge(a,b)$	$\max\{0, a+b-1\}$	$\min\{a,b\}$	$a\cdot b$	$a \cdot b$
	$\overline{a}$	$\overline{a}$	$\overline{a}$	$\overline{a}$
NOT a	$(1-a)$	$(1-a)$	$(1-a)$	$(1-a)$

The *global function* of the CA

$$\underline{g} : S^Z \to S^Z$$

associates to any configuration $\underline{x} \in S^Z$, the configuration at the next time step: $\underline{g}(\underline{x}) = (\ldots, g_{i-1}(\underline{x}), g_i(\underline{x}), g_{i+1}(\underline{x}), \ldots) \in S^Z$; where, $\forall i \in Z$, its i-component $g_i : S^Z \to S$ specifies the next state of site i according to the following rule:

$$\forall i \in Z \quad g_i(\underline{x}) := h(x_{i-r}, x_{i-r+1}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+r-1}, x_{i+r}).$$

The local function h is a function between fuzzy values, and its nature will be discussed in detail later.

Notice that a fuzzy CA could be alternatively defined as a coupled map lattice [12] where the local function is the "fuzzification" of a CA rule.

In the following, we shall consider elementary fuzzy CAs, i.e., one-dimensional, bi-infinite CAs with unitary radius, $r = 1$. In particular, we shall consider elementary fuzzy CAs in which the initial configuration has a finite number of cells with a boolean state, embedded in a fuzzy background (*CAs with fuzzy background*); that is, the initial configuration has the following form:

$$\underline{x}(0) = (\ldots, x_{-m-1}, x_{-m}, \ldots, x_{-1}, x_0, x_{+1}, \ldots, x_m, x_{m+1}, \ldots),$$

where

$$\begin{cases} x_i \in \{0,1\} & \text{if } -m \le i \le +m, \\ x_i \in S & \text{otherwise.} \end{cases}$$

For simplicity of notation we assume that the boolean string consists of an odd number of cells; the case of an even number is treated using a similar notation.

2.2. *Fuzzy operators*

The local function $h : S^3 \mapsto S$ is built starting from the disjunctive normal form $f : \{0,1\}^3 \mapsto \{0,1\}$ of a boolean CA rule with a "fuzzification" process, that is, using a fuzzy extension of the boolean operators AND, OR and NOT; depending on which fuzzy operators will be used a different class of fuzzy CA will be defined. Table 1 shows some possible choices of fuzzy operators [7].

Given a boolean CA with local rule f, we want to construct the fuzzy local rule h of the corresponding fuzzy CA; clearly, the fuzzy local rule h must coincide with f when the states are boolean. For this "fuzzification" we have chosen the following fuzzy operators: the AND operator corresponds to the *product* (i.e., $\wedge(a,b) = a \cdot b$), the OR corresponds to the fuzzy MV operator [7] (i.e., $\vee(a,b) = \min\{1,(a+b)\}$) and the NOT operator corresponds to the complement (i.e., $\overline{a} = (1-a)$).

We define, for every $x \in S$ and $a \in \{0, 1\}$,

$$x^a = \begin{cases} x & \text{if } a = 1, \\ (1-x) & \text{if } a = 0. \end{cases}$$

It is easy to see that, using the fuzzy operators described above, the local rule h becomes

$$h(x_{i-1}, x_i, x_{i+1}) = \sum_{a,b,c \in \{0,1\}} x_{i-1}^a \cdot x_i^b \cdot x_{i+1}^c \cdot f(a, b, c).$$

Example 1. Consider, for example, the CA rule 18. In this case the local rule can be represented by the following table:

x_{i-1}	x_i	x_{i+1}	$f(x_{i-1}, x_i, x_{i+1})$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

The boolean expression of this rule is $f(x_{i-1}, x_i, x_{i+1}) = (\overline{x_{i-1}} \wedge \overline{x_i} \wedge x_{i+1}) \vee (x_{i-1} \wedge \overline{x_i} \wedge \overline{x_{i+1}})$. By the fuzzification process, we have that the corresponding fuzzy rule is the following:

$$h(x_{i-1}, x_i, x_{i+1}) = (x_{i-1}^0 \cdot x_i^0 \cdot x_{i+1}) + (x_{i-1} \cdot x_i^0 \cdot x_{i+1}^0).$$

2.3. Windows of observability and behavior

Since our purpose is to understand the real influence of the fuzzy background on the dynamics of a CA, we consider fuzzy initial configurations with a certain window of "boolean" values (*boolean string*) and study the interaction between fuzziness and boolean cells.

Different modalities can be used for building the initial fuzziness of the background. Three are of particular interest: *random* background, in which the states of the cells are randomly chosen in [0,1]; *smoothing* background, where fuzziness which is close to 0 and 1 near the boolean window goes to $\frac{1}{2}$ moving towards infinity; and *homogeneous* background in which each cell is in the fuzzy state $\frac{1}{2}$ (*homogeneous background*).

We have considered all three environments. Interestingly, the results we have obtained are not affected by the choice of the background and are thus independent of the modality.

Instead of observing the evolution of CAs in a fixed-size window, we consider an expanding window in which it is possible to distinguish two regions: a *boolean window* where only the evolution of the boolean cells is observable and an *interaction window* in which the fuzzy background interacts with the boolean cells.

Given the initial configuration $\underline{x}$ with a boolean string of length $n = 2m + 1$, the *boolean window* ($BW_n(\underline{x})$) is the set of strings

$$BW_n(\underline{x}) = \{(x^t_{-m+t}, \ldots, x_{m-1_t}) : t = 0, \ldots, m\}.$$

Similarly, given the initial configuration $\underline{x}$ with a boolean string of length $n = 2m + 1$, the *interaction window* ($IW_n(\underline{x})$) is the set of strings

$$
\begin{aligned}
IW_n(\underline{x}) = \{&(x^t_{-m-t}, \ldots, x^t_{m+t}), (x^t_{m-t}, \ldots, x^t_{m+t}) : 0 \le t \le m\}. \\
&\cup \{(x^t_{-m-t}, \ldots, x^t_{m+t}) : t > m\}.
\end{aligned}
$$

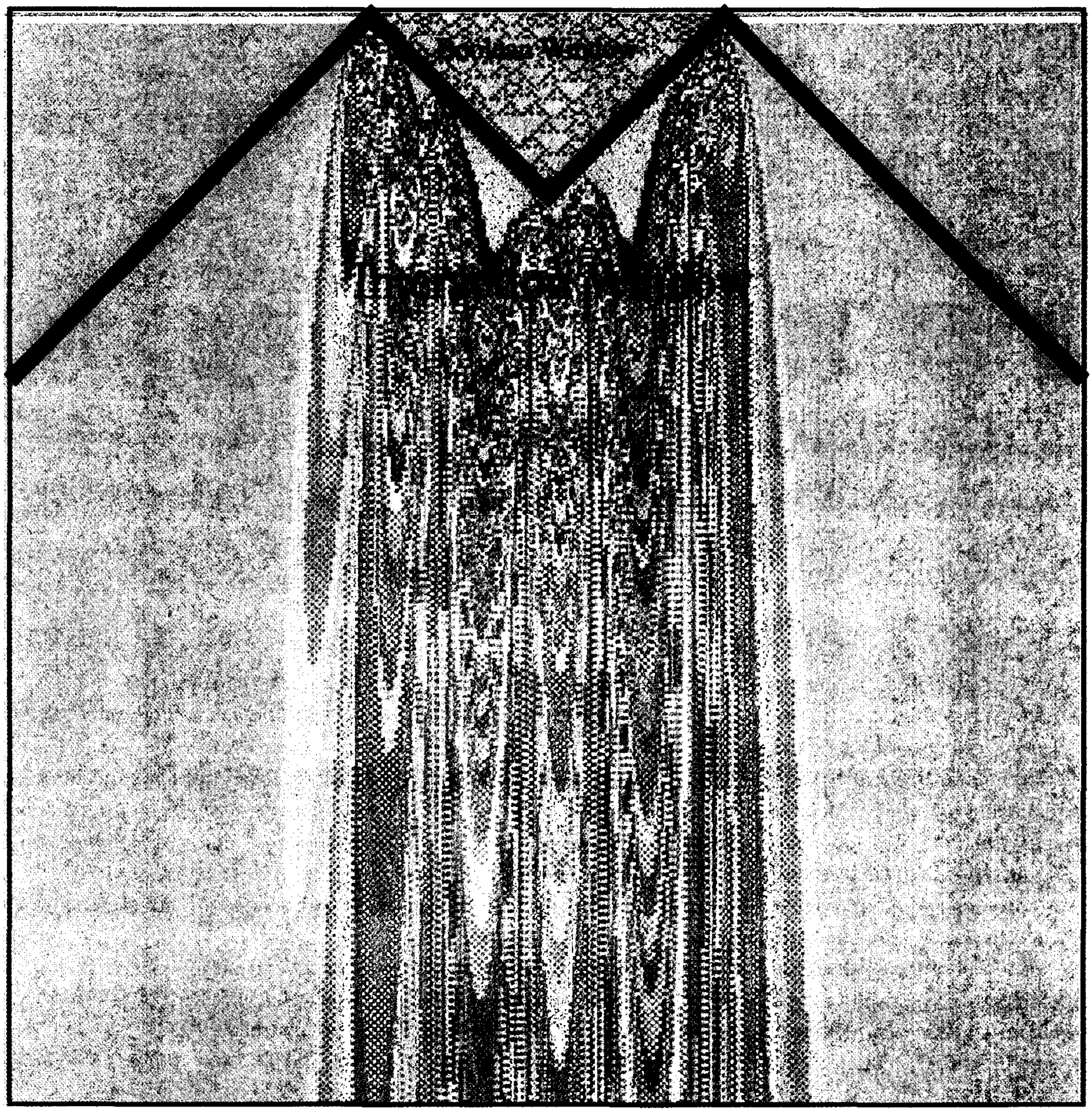


Fig. 1. Boolean and interaction windows in space–time patterns generated by fuzzy CAs.

Fig. 1 shows the boolean window and the interaction window on a space–time pattern; observe that in the boolean window $BW_n(\underline{x})$ we actually observe the evolution of the boolean CAs. In this and all subsequent figures, the initial background is homogeneously fuzzy.

Our interest is in the interaction window, the region in which the fuzziness of the background interacts with the boolean values. The interaction between fuzziness and boolean cells provides information which cannot be detected by the classical discrete model of finite CAs. The dynamics of the elementary CAs in fuzzy background partitions them into three classes depending on their behaviors within the interaction window:

(1) *Boolean behavior.* The initial boolean string prevails over the fuzziness of the background; after a finite number of steps, all the cells in the interaction window $IW_n(\underline{x})$ are boolean.
(2) *Boolean/fuzzy behavior.* The interaction window contains both boolean and fuzzy values; after a finite number of steps, $IW_n(\underline{x})$ is partitioned into boolean and fuzzy subregions.
(3) *Fuzzy behavior.* The interaction window contains only fuzzy values; that is, the fuzziness of the background destroys the boolean values, and after a finite number of steps, all the cells have fuzzy values. Within this type of behavior we can distinguish two different forms of propagation of fuzziness:
   (a) *homogeneous fuzziness* – all the cells assume the same fuzzy values;
   (b) *heterogeneous fuzziness* – the states of the cells can assume different fuzzy values.

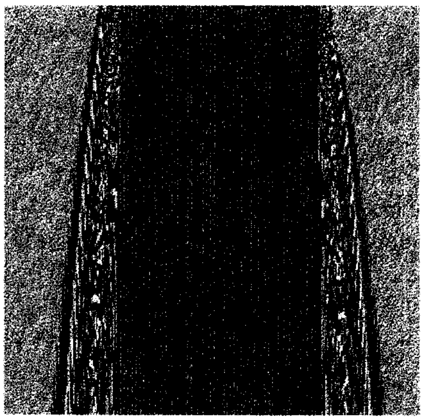
3. Behaviors: Empirical evidence

We have observed the evolution of the 256 elementary fuzzy CAs initialized in configuration with homogeneous fuzzy background. The observed behavior of these CAs provides empirical evidence for the partition discussed below. An interesting aspect of this partition is that CAs which, according to Wolfram's classification, belong to the same class have drastically different observable behaviors.

In Table 2 some examples of these different behaviors are shown.

Boolean behavior. $IW_n(\underline{x})$ is boolean. The fuzziness of the background does not affect the evolution of the automata; on the contrary, the boolean values given by the initial configuration propagate outside the boolean window BW_n (see rule 87 in Fig. 2). Not surprisingly, this type of behavior is only found among the simplest CAs, belonging to Wolfram's Classes 1 and 2.

Rule 87



Rule 133

Fig. 2. Different "fuzzy" behavior of rules belonging to the same boolean Wolfram class. Rule 87 is an "odd" rule and rule 133 a 90-like rule.

Table 2

Behavior	Rules		Wolfram's classification
Boolean	0, 3, 7, 8, 32, 64, 128, 40, 136, 160, 168		
	36, 72, 104, 164, 200, 234		Class 1
	1,5		Class 2
Boolean/fuzzy	12, 13, 77, 132, 140, 232		
	4, 44, 76, 78, 172, 204, 15, 42, 170		Class 1
	2, 10, 24, 34, 130, 138, 152, 162, 188		
	23, 28, 29, 50, 51, 73, 94, 108, 156, 178, 199		Class 2
	Homogeneous	Heterogeneous	
Fuzzy	6, 9, 25, 35, 38, 57	14, 27, 43, 56, 58, 74	
	61, 62	134, 142, 184	Class 1
	37		Class 2
	22, 26, 30, 41, 45, 54, 60, 90	18, 126, 146	Class 3
	105, 106, 110, 122, 150, 154		

Boolean/fuzzy behavior. $IW_n(\underline{x})$ contains fuzzy and boolean values. Again, this type of behavior is exhibited solely by CAs belonging to Wolfram's Classes 1 and 2. The fuzziness of the background does not influence the evolution of the automata in its boolean window. The region containing boolean values has a fixed length (see rule 133 in Fig. 2).

Fuzzy behavior. After a finite number of steps, the interaction window $IW_n(\underline{x})$ contains only fuzzy values. The fuzziness of the background propagates inside the boolean window and destroys the boolean values. This is the most interesting situation in which complex behavior seems to appear (see rules 143, 115, 126 and 185 in Fig. 3).

This behavior is exhibited by all the rules belonging to Wolfram's Class 3 and considered as "chaotic". However, it is also shown by some "simple" rules of Classes 1 and 2.

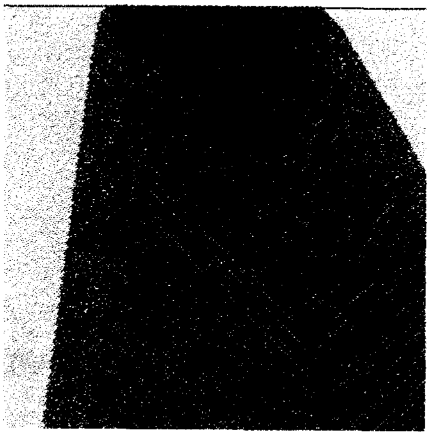
As an example consider rule 184 (see Fig. 5). The corresponding CA belongs to Class 1 and is generally considered one of the simplest. However, its evolution in fuzzy backgrounds indicates otherwise ("it shows a different picture"); in particular the interaction between fuzziness and boolean cells creates complex structures. Even though it is difficult to theoretically explain this behavior, the experimental results quite clearly indicate that this is not a simple rule.

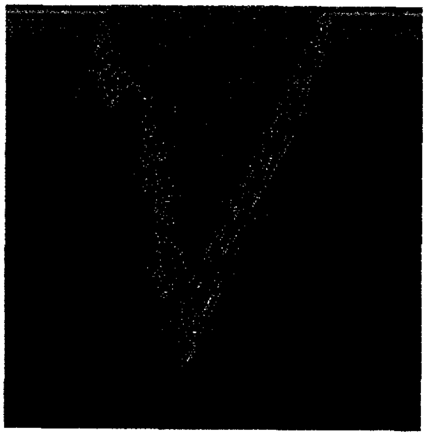
As mentioned before, there are two types of fuzzy behaviors:

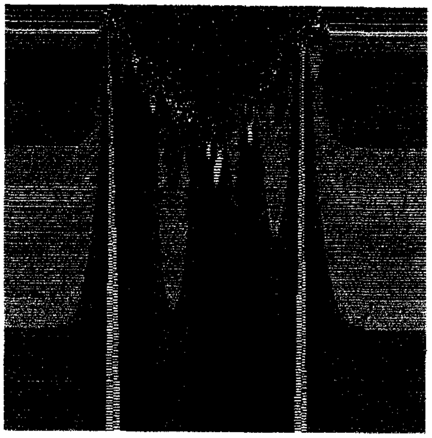
(1) *Homogeneous fuzziness* – This behavior is exhibited, for example by rule 90 (Fig. 4). After a certain number k of steps, all the cells assume the same fuzzy value. In this case, the observation of a configuration after step k does not give any information about the preceding dynamics of the automaton nor about the CA itself.

(2) *Heterogeneous fuzziness* – This behavior is exhibited, for example by rule 18 (Fig. 4). All cells have fuzzy values, but these values are not the same (though, they appear to be correlated). In this case, the fuzzy values seem to follow a particular, fuzzy distribution and they create interesting patterns.

These two situations describe quite different dynamics. They allow to see a clear distinction between rules which were considered equal. For example, the heterogeneous fuzziness reveals difference in the dynamics of rules 184 and 57. Both rules are shifting rules; when observed in a quiescent background, they belong to Class 1 and have exactly the same dynamics. In the evolution of rule 57, the fuzziness propagates homogeneously in the boolean region. After a finite number of steps all the cells have the same fuzzy value and no information is left about the preceding evolution; while for rule 184, the fuzziness propagation forms complex patterns whose structure can be seen as a "signature" of the CAs (see Fig. 5); in other words, some information about the preceding evolution is maintained.


Rule 143


Rule 115


Rule 126

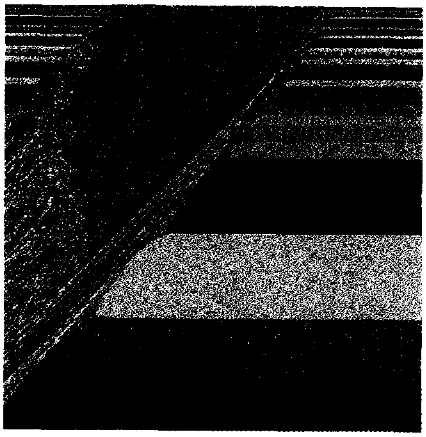

Rule 185

Fig. 3. Fuzzy behavior: rules in which the fuzziness of the background propagates inside the boolean window. Both are boolean sub-shifts.

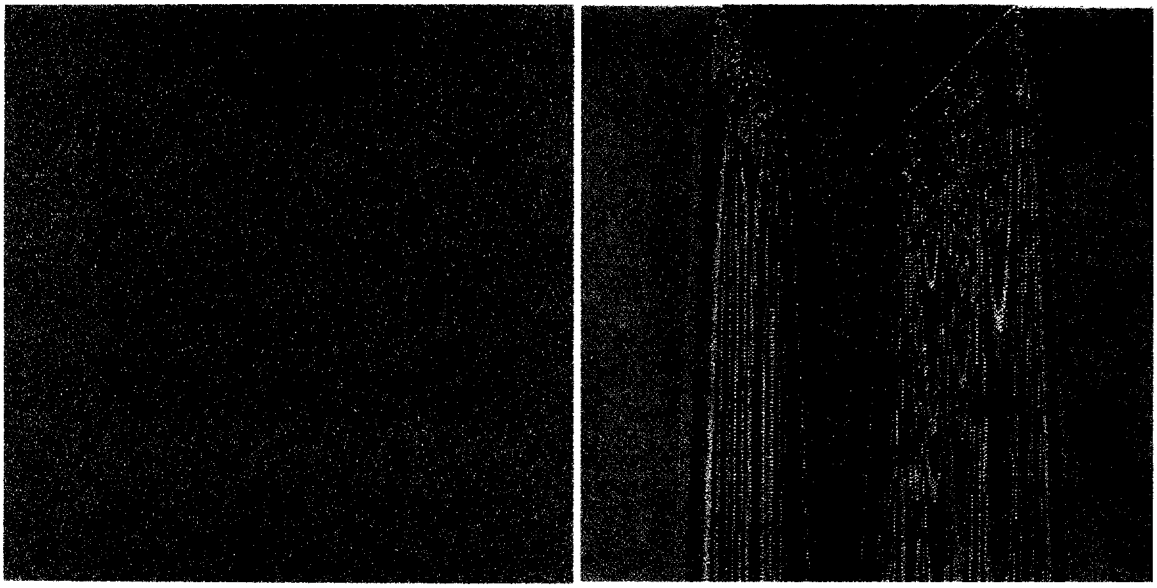
We suggest that the "chaotic" CAs are those automata showing heterogeneous fuzziness, that is, in which the interaction between the background and the boolean window creates complex patterns.

Our purpose is to study the fuzzy behavior, trying to understand which are the characteristics that allow to distinguish these automata from the others and to analyze the shape and the patterns which emerge from the background.

In Section 4, we introduce a new measure, the *rule entropy*, which explains some of the observed phenomena and provides support for our interpretations.

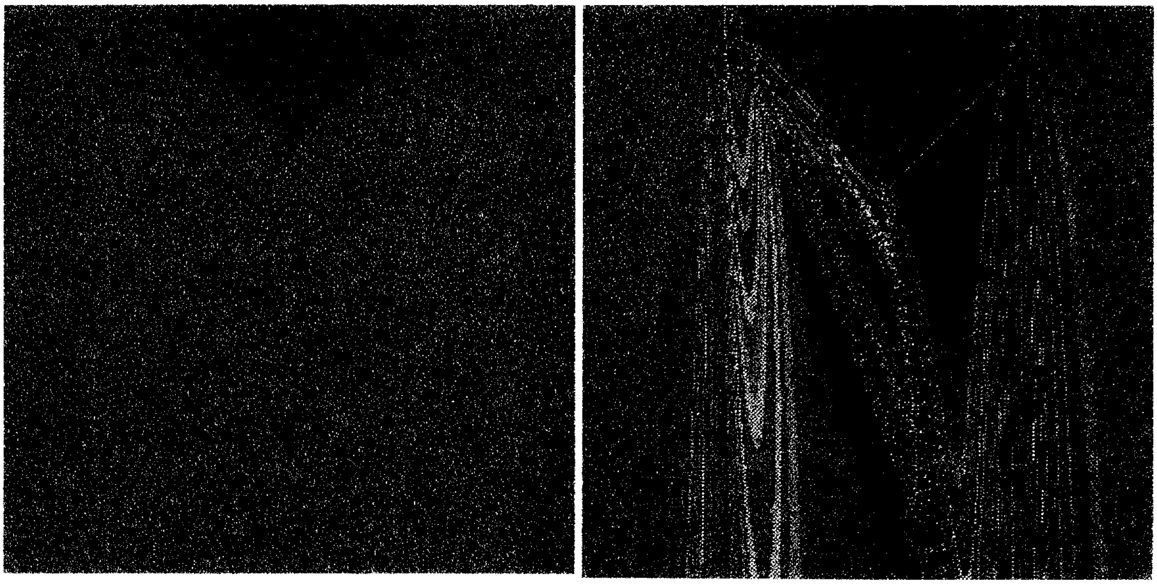
4. Fuzzy CAs and rule entropy

As experimentally observed, some CAs behave in a homogeneous and very regular way in the interaction window; on the other hand, for some CAs, the interaction window is highly irregular. Different interactions correspond to



rule 90 rule 18

Fig. 4. Rules with fuzzy bahavior: homogeneous fuzziness (rule 90) and heterogeneous fuzziness (rule 18). This latter is a 90-like rule



rule 57 rule 184

Fig. 5. Difference of fuzzy behavior of boolean shifting rules.

different dynamics, at least in the length of the transient; in particular, heterogeneous fuzzy behavior indicates complex dynamics. To understand this difference, it is useful to interpret a boolean value as a known value, and a fuzzy value as a value known with uncertainty. In this light, the evolution in the interaction window of a CA represents its behavior in the region where certainty and uncertainty interact. Thus, the different dynamics in this region could be captured by some measures of the propagation of uncertainty. In the following we introduce a particular form of entropy (*rule entropy*) which provides a quantitative description of the interaction; this description supports several of the experimental observations.

4.1. Rule entropy

The rule entropy (RE) is a measure strictly related to the rule of a CA and is *totally independent* of the choice of the initial configuration. Unlike other forms of entropies for CAs proposed in the literature (e.g., [15,16,20,22]), RE is a measure solely of the rule structure and not of the configuration. Furthermore, it is effectively computable since it only assumes values in a finite (small) set. Informally, it measures how much the initial uncertainty on the values of some cells influences the knowledge about the future configurations of the CA and propagates during its evolution. In other words, the RE expresses the inherent tendency of a CA to increase disorder in presence of uncertainty.

The triple $(\{0,1\}, \mathcal{P}(\{0,1\}), \mu_c)$ is a probability space, where μ_c is the count measure. A CA rule f can be viewed as a boolean random variable on the phase space $\{0,1\}^3$. Let $x, y \in \{0,1\}$ be fixed; let $f_{xy}^l : \{0,1\} \to \{0,1\}$ be the map $f_{xy}^l(t) := f(t, x, y)$. On the probability space defined above, we introduce the partition:

$$\alpha_{f_{xy}^l} := \{A_0, A_1\},$$

where

$$A_0 = f_{xy}^{l-1}(0), \quad A_1 = f_{xy}^{l-1}(1).$$

A_0 (resp. A_1) can be seen as the event "a measurement of the random variable f for x, y fixed give the value 0 (resp. 1)". We have

$$\mu_c(A_1) = \tfrac{1}{2}[f(0,x,y) + f(1,x,y)],$$

which represents the probability that the application of the rule with x, y fixed, respectively, in the second and third position, yields the value 1. Similarly, we have that

$$\mu_c(A_0) = 1 - \tfrac{1}{2}[f(0,x,y) + f(1,x,y)].$$

Now we can calculate the entropy of the partition $\alpha_{f_{xy}^l}$ using the canonical definition of entropy for a partition:

$$H\left(\alpha_{f_{xy}^l}\right) = \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \log \frac{2}{\sum_{a\in\{0,1\}} f(a,x,y)} + \left(1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2}\right) \log \frac{1}{1 - \sum_{a\in\{0,1\}} f(a,x,y)/2}.$$

Table 3
RE of some rules

Rule	E_{left}	E_{right}	Behavior
90	6	6	Fuzzy (homogeneous)
18	3	3	Fuzzy (heterogeneous)
15	6	0	Boolean/fuzzy (one-side) (homogeneous)
112	4.5	1.5	Boolean/fuzzy (one-side) (hetrogeneous)
57	4	4	Fuzzy (almost homogeneous)
184	3	3	Fuzzy (heterogeneous)
240	6	2	Fuzzy (homogeneous)
56	4.5	2.5	Fuzzy (heterogeneous)
64	1.5	1.5	Boolean
37	4.5	4.5	Fuzzy (heterogeneous)

For different values of the parameters x and y we obtain different partitions and then different entropies; if we sum all these quantities, we obtain the *left*-1 *RE*:

$$H_l^{(1)} = \sum_{x,y\in\{0,1\}} \left(\sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \right) \log \left(\frac{2}{\sum_{a\in\{0,1\}} f(a,x,y)} \right) + \left(1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \right) \log \frac{1}{1 - \sum_{a\in\{0,1\}} f(a,x,y)/2}.$$

Fixing only a value z we can calculate in a similar way the left- 2 RE; and, analogously, we can define *right*-1 and right-2 REs.

In the following, we shall call left-RE the quantity $E_{\text{left}} = H_l^{(1)} + H_l^{(2)}$ and right-RE the quantity $E_{\text{right}} = H_r^{(1)} + H_r^{(2)}$.

4.2. Rule entropy and fuzziness

Giving to fuzziness the meaning of uncertainty, the evolution in the interaction window of a CA represents its behavior in a region where certainty and uncertainty interact. Since the RE is related to the amount of information that CA evolution generates when some cells are unknown, this measure reflects in some way the dynamics of the interaction window.

In a CA where the boolean string of the initial configuration is completely destroyed by the background (homogeneous fuzziness) we expect a high RE, expressing uncertainty and disorder. On the other hand, we expect an automaton in which the boolean values propagated in the interaction window to have a zero RE, that is, absence of disorder.

The RE has been computed for all elementary CAs; in Table 3 some numerical values of RE are given. The numerical values do confirm the expectations raised by the experimental observations using fuzzy backgrounds. In particular, RE separates the homogeneous and the heterogeneous fuzzy behavior, and distinguishes CAs which, observed in the traditional quiescent background, appeared to have similar dynamics. In the following, an example is given.

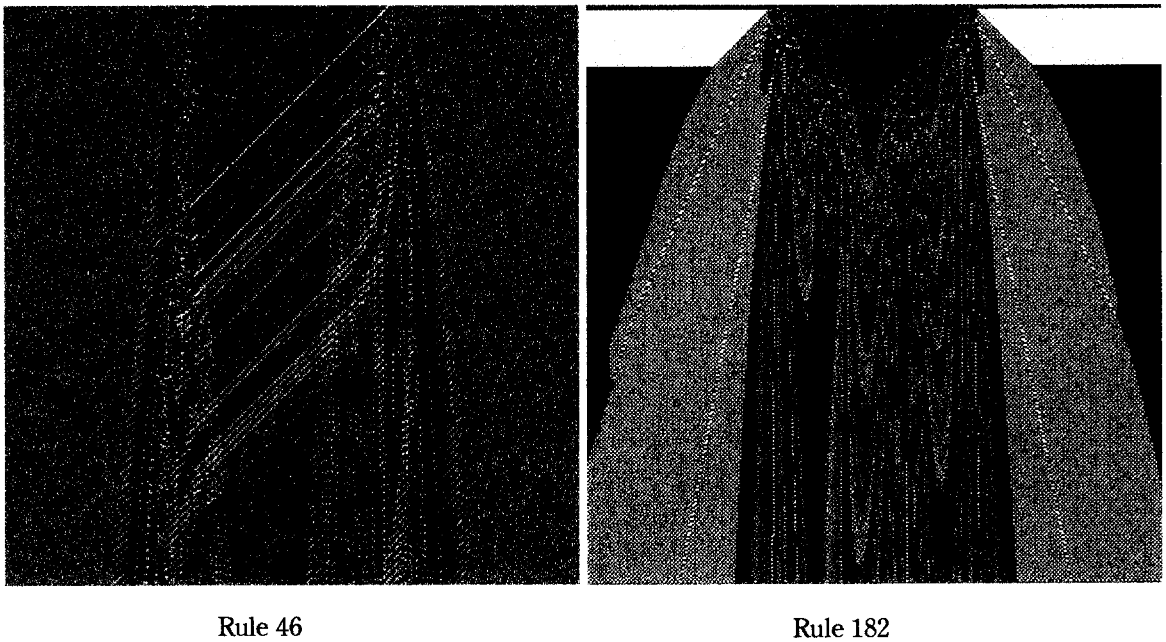


Fig. 6. Complex patterns generated by boolean sub-shift rule (46) and a 90-like boolean rule (182). Both showing partial heterogeneous fuzziness.

Example 2. Consider rules F_{90} and F_{18} (see Fig. 4). In both cases, the boolean region is destroyed by the fuzziness, and the interaction window has only fuzzy values; however, the fuzziness propagates in very different ways for the two rules. During the evolution of rule F_{90}, after a finite number of steps, all cells have value $\frac{1}{2}$ (maximum of uncertainty). In the evolution of rule F_{18}, the boolean region disappears after a finite number of steps, but fuzziness propagates forming complex patterns (heterogeneous fuzziness). The difference in the dynamics of the two rules is detected by the RE. In fact, for rule F_{90} RE assumes the maximum (left and right) value ($E_{\text{left}} = 6$, $E_{\text{right}} = 6$) reflecting the propagation of disorder, while for rule F_{18} it has an intermediate value ($E_{\text{left}} = 3$, $E_{\text{right}} = 3$).

It is interesting to note that, in some CAs, the fuzziness appears only in one side of the interaction window. The left and right entropies capture the difference not only between homogeneous and heterogeneous, but also between *one-sided* and global fuzziness, as well as the type of fuzziness.

Example 3. Consider rules F_{240} and F_{56} (Fig. 7). Both rules have a boolean/fuzzy behavior; however, the propagation of fuzziness is quite different. Rule F_{240} is a right shift; during the evolution the fuzziness propagates homogeneously on the left, destroying all the boolean values and the boolean values propagate on the right at maximum speed. This behavior is captured by the fact that the left-entropy is maximum ($E_{\text{left}} = 6$) and the right-entropy is minimum ($E_{\text{right}} = 0$). Also rule F_{56} is a right shift, but the interaction between the boolean and the fuzzy region is not as defined as in the previous case; it is more complex, and it reveals chaotic structures; this is captured by intermediate values of RE ($E_{\text{left}} = 4.5$ and $E_{\text{right}} = 3.5$).

It should be stressed that, while the RE does not provide a complete characterization of the elementary CA, it gives useful information about the dynamics in the interaction window.

The empirical evidence shows that the RE allows to distinguish between certain "chaotic" and "nonchaotic" behaviors; unfortunately, this measure alone is not able to describe the dynamics of CAs in the "Phase Transition".

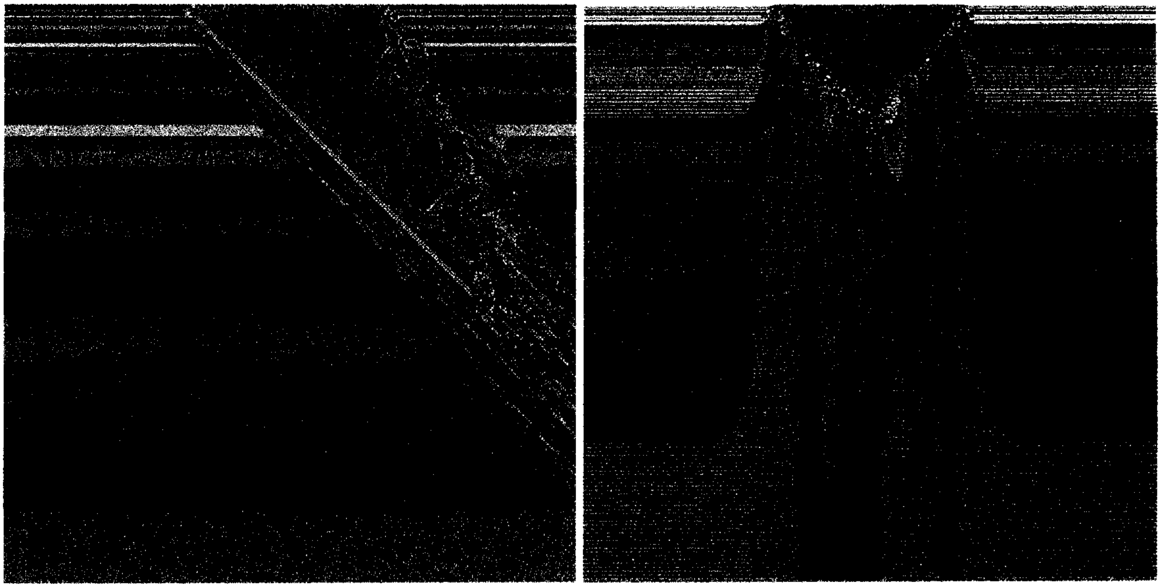


Fig. 7. Rule 56: boolean shift behavior with intermediate values of RE (4.5 and 3.5). Rule 129 is another 90-like boolean rule. Both showing complete heterogeneous fuzziness.

5. Fuzzy CAs and linearity

In this section, we will discuss the relationship between the experimental results, the RE and the linearity of the CAs. Linearity is often a sign of simple dynamics, and chaotic behavior always appears when a system has a nonlinear description.

Let us introduce some terminology.

A rule f is *independent* of x_{i-1} iff $\forall x_i, x_{i+1} \in \{0,1\}$, $f(0, x_i, x_{i+1}) = f(1, x_i, x_{i+1})$. It *depends linearily* on x_{i-1} iff $\forall x_i, x_{i+1} \in \{0,1\}$, $f(0, x_i, x_{i+1}) \neq f(1, x_i, x_{i+1})$ the corresponding CAs will be called *left-linear* CAs. A rule f *depends nonlinearly* on x_{i-1} iff $\exists\, x_i, x_{i+1} \in \{0,1\}$ such that $f(0, x_i, x_{i+1}) = f(1, x_i, x_{i+1})$ and $\exists\, x_i, x_{i+1} \in \{0,1\}$ such that $f(0, x_i, x_{i+1}) \neq f(1, x_i, x_{i+1})$. The corresponding CAs will be called *left-nonlinear* CAs.

In a similar way we can define *right-linear*, *center-linear*, *right-nonlinear* and *center-nonlinear* CAs.

Linearity is a well-known property (e.g. [22]), sometimes called permutivity [16] or injectivity [10].

The RE captures the relationship between linearity of the rules and complex dynamics of CAs. In fact, there is a strict relationship between the linearity of a rule, its entropy, and the structures of the background. In particular:

Proposition 1. A left/right-linear rule f has the maximum left/right-rule entropy, while a rule with no dependency on the left/right has zero left/right rule entropy.

Proof. Consider a left-linear rule f. By definition of linearity it is $f(0,x,y) \neq f(1,x,y), \forall x, y \in \{0,1\}$, that means $f^l_{xy}(0) \neq f^l_{xy}(1)$. It follows that $\mu_c(A_1) = \mu_c(A_0) = \frac{1}{2}$ and thus, $\forall x, y \in \{0,1\}$, $H(\alpha_{f^l_{xy}}) = 1$. Thus, $H_l^{(1)} = \sum_{x,y,\in\{0,1\}} H(\alpha_{f^l_{xy}}) = 4$, that is, the maximum possible value for $H_l^{(1)}$.

By definition of linearity it is $f_z^l(00) \neq f_z^l(01)$ and $f_z^l(11) \neq f_z^l(10)$. It follows that $\mu_c(A_1) = \sum_{a,b\in\{0,1\}} f(abz)/4 = \frac{1}{2}$, thus $H(\alpha_{f_z^l}) = 1 \forall z \in \{0,1\}$ and thus $H_l^{(2)} = \sum_{z\in\{0,1\}} H(\alpha_{f_z^l}) = 2$ that is the maximum possible value for $H_l^{(2)}$. □

An intermediate value of the RE is characteristic of nonlinear rules with different levels of linearity. Based on these results, the RE can be considered a measure, not only of disorder, but also of the degree of linearity of a CA rule. This fact gives more strength to the conjectured existence of a link between nonlinearity, disorder, and complexity. For having "chaotic" behavior the simultaneous presence of both order and disorder is necessary. In fact, the rules whose entropy is maximum (i.e., where disorder propagates) and the rules with zero entropy (i.e., where only order propagates) have very simple dynamics.

In other words, chaos and complex dynamics can appear only in intermediate situations, when the dynamics has a component of regularity and a component of disorder. Let us stress that an intermediate value of RE is a necessary, but not sufficient condition for having a complex fuzzy behavior in CAs.

We suggest as interesting argument the investigation of these situations of nonlinear rules characterized by intermediate entropy values.

6. Conclusions

We have defined a model of fuzzy CAs which provides a new tool for the understanding of their complex dynamics.

Fuzzy CAs are CAs in which a finite boolean configuration evolves in a background of fuzzy values; in this context a CA is considered to have complex dynamics when the interaction between the boolean evolution of the CA and the background creates complex structures and chaotic patterns.

We have proposed an empirical classification of CAs on the basis of their behavior in fuzzy backgrounds and we have found some theoretical supports to our observations.

The fuzzy model allows to detect the inherent difference between rules which are considered equal in the literature. Two significant examples are represented by the rules 90, 18 and rules 57, 184. Consider for example the case of CA rules 18 and 90 (Fig. 4); when they evolve in quiescent backgrounds or with circular configurations, they have the same triangular structures typical of chaotic behavior. Their dynamics seem to be exactly the same even if rule 90 is a linear rule and rule 18 is nonlinear; thus the two CAs are both considered "chaotic" and they belong to Class 3 in Wolfram classification. The fuzzy background allows to distinguish between the two dynamics suggesting that the two CAs must have very different natures. When observed in a fuzzy background, rule 18 has a complex behavior, while rule 90 seems very simple, becoming totally homogeneous after a finite number of steps.

An easily computable measure which captures several of the observed differences in CA's dynamics has been defined. This measure (rule entropy) reflects in some way the dynamics of the interaction window giving a theoretical support to some of the experimental results.

These results open interesting research directions including, for instance: the analytical treatment of the different dynamics uncovered here and the identification of measures (e.g., different entropies) capable of characterizing the "phase transition" between the classes we have identified. An important open question is whether other generalizations of CA to nonbinary inputs are possible or natural.

Of particular interest, in the case of "pure" fuzzy CAs (i.e., without boolean values), is the characterization of those systems for which any non-boolean initial configuration would converge towards a boolean one.

Another interesting open question is the determination of the "asymptotic number of states" taken by the different fuzzyfied rules.

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